Counting Saddle Connections on Slit Translation Surfaces

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Brooklyn College and The Graduate Center (CUNY)

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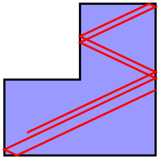
Billiards in a polygon

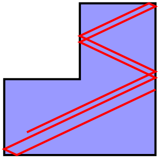


Billiard

Setup:

- Consider a point particle moving in a polygon with elastic reflections.
- At each boundary hit, the angle of incidence equals the angle of reflection.
- Assume no friction, mass or spin (like a photon!)

Unfolding a Polygonal Billiard



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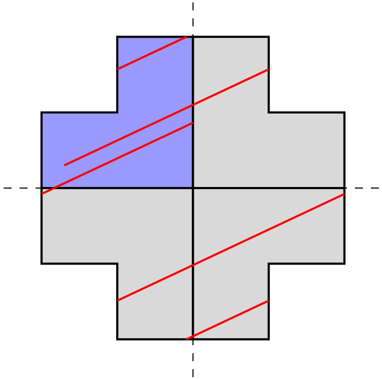

Unfolded billiard

Unfolding Idea:

- Instead of reflecting the trajectory, reflect the *table*.
- Each time the particle hits an edge, glue a reflected copy of the polygon along that edge.
- The trajectory becomes a straight line across multiple copies.

Unfolding a Polygonal Billiard

Resulting Structure:

- Instead of reflecting forever, anytime a reflection already occurred, identify the sides.
- The billiard flow corresponds to straight-line motion on this surface.
- Identifying parallel edges yields a *translation surface*.

Key Insight:

- Complicated reflections $\rightarrow$ straight-line dynamics.
- Dynamical questions reduce to geometry of the unfolded surface.

Unfolding converts billiard trajectories into geodesics.

Translation Surfaces

- Let (X, ω) be a translation surface (Riemann surface of genus $g \geq 1$ carrying a holomorphic Abelian differential)
- These can be thought of as polygons with opposite sides identified and a choice of direction
- The vertices of the polygon have cone angles that are multiples of 2π corresponding to zeros of the Abelian differential

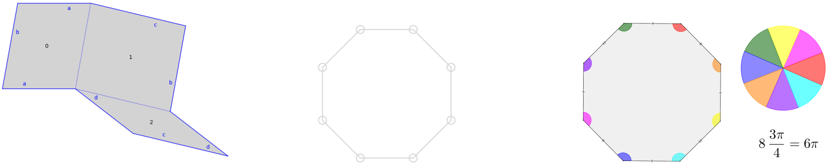

Figure: Translation Surfaces and Cone Angles

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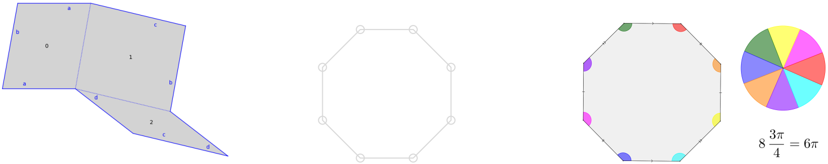

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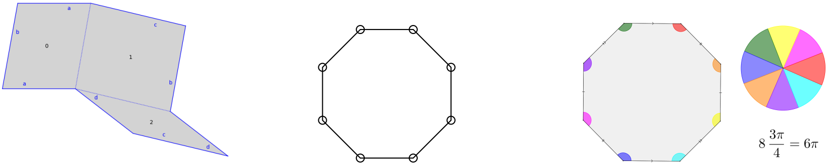
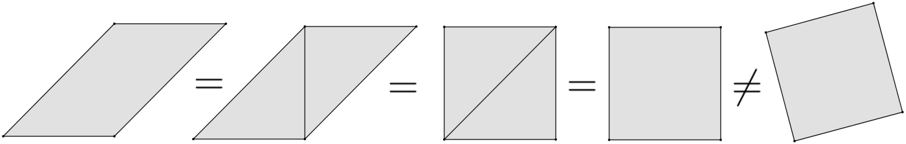


Figure: Translation Surfaces and Cone Angles

Strata of Translation Surfaces

- Let $\mathcal{H}(\kappa)$ be a stratum of translation surfaces: A moduli space of translation surfaces with specified cone angles
- The total order of the zeros of the Abelian differential are $2g - 2$, so κ is a partition of $2g - 2$
- Strata admit finite Masur-Veech measures (absolutely continuous with respect to Lebesgue)



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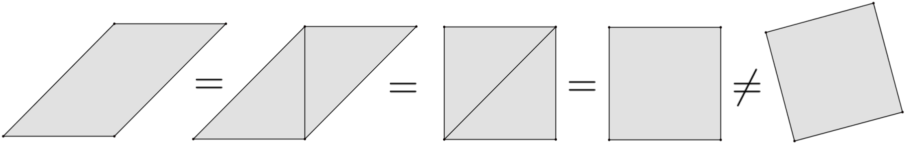

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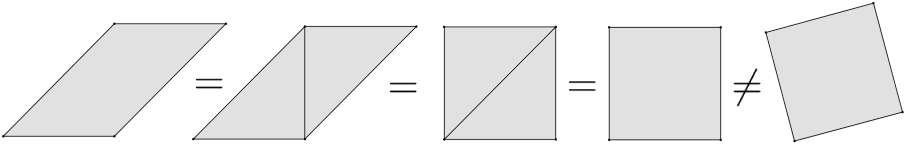
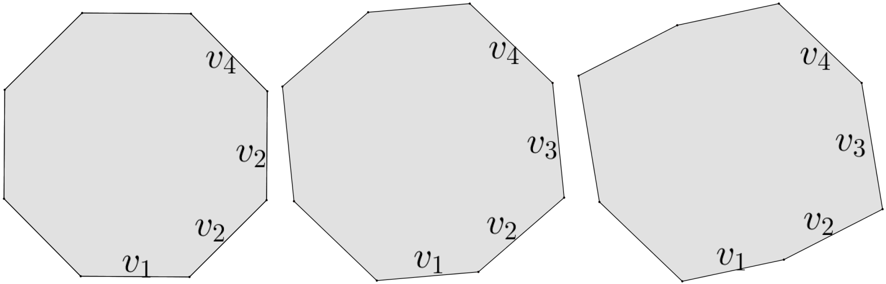


Period Coordinates

Definition

Strata admit local charts given by *period coordinates* in a neighborhood of a translation surface (X, ω). Let Σ be the zeros of ω. Let $\{\gamma_1, \ldots, \gamma_n\}$ be a basis for $H_1(X, \Sigma, \mathbb{Z})$. We have the map

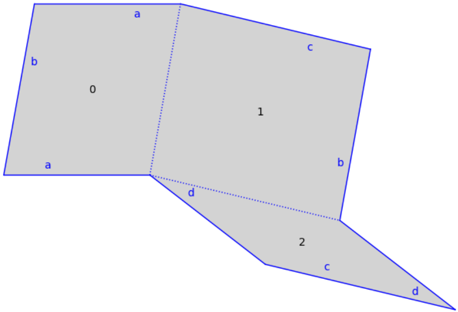
$$\Phi(X, \omega) = \left(\int_{\gamma_1} \omega, \ldots, \int_{\gamma_n} \omega \right) \in \mathbb{C}^n.$$



Connected Components of Strata

Strata are *not* always connected

- Classification [Kontsevich-Zorich]: There are at most three connected components.
- If $\kappa \in \{(2g-2), (g-1, g-1)\}$, then one component is *hyperelliptic*.
- The component $\mathcal{H}^{hyp}(\kappa)$ is a full-dimensional locus of translation surfaces that admit a hyperelliptic involution.



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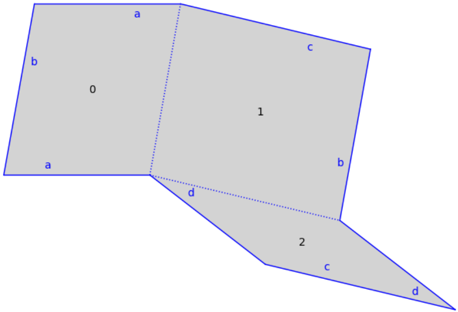

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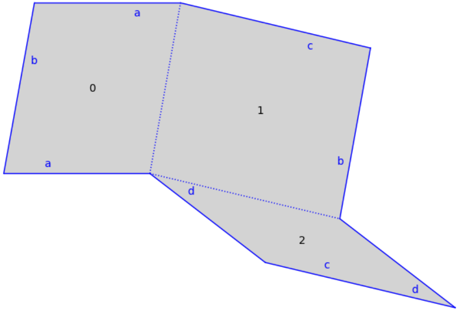

$GL_2^+(\mathbb{R})$ Action on Strata

There is a natural action by $GL_2^+(\mathbb{R})$

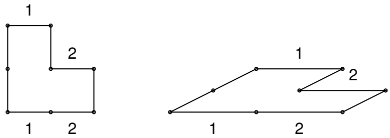

Figure: Action by $\begin{pmatrix} 2 & 1 \\ 0 & 1/2 \end{pmatrix}$

- The *Teichmüller flow* is the diagonal action with determinant 1

$$\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

- The *horocycle flow* is the unipotent action

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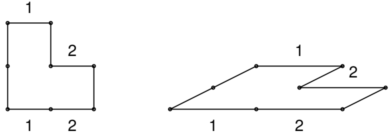

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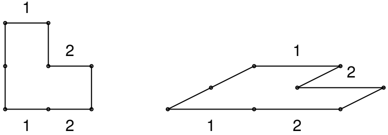

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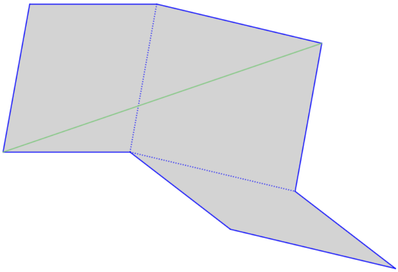
Saddle Connections

- Translation surfaces have natural straight line flows
- We can consider special (finite) trajectories that start and end at a singularity of ω - *saddle connections*
- The holonomy vector of a saddle connection is a vector in $\mathbb{R}^2$
- Saddle connections have natural flat lengths



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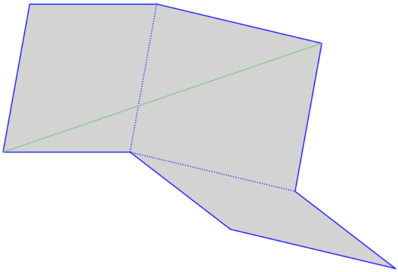

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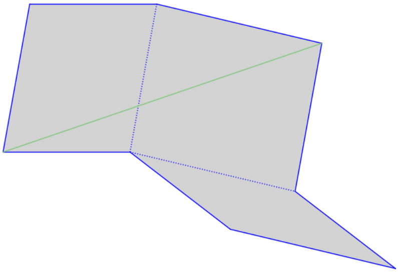

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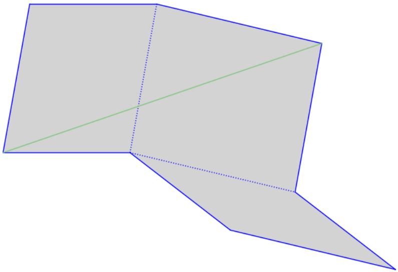

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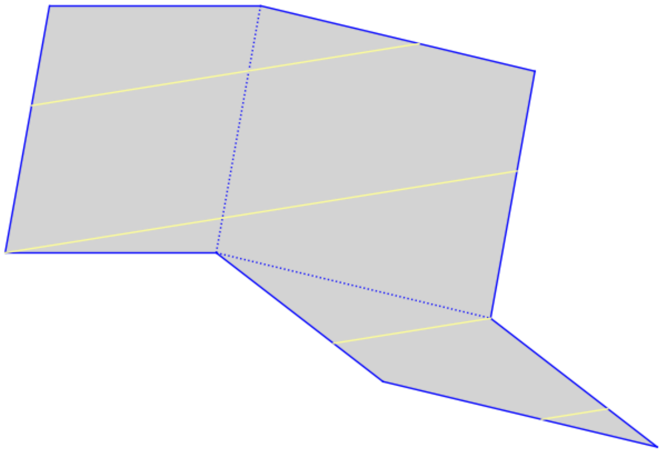

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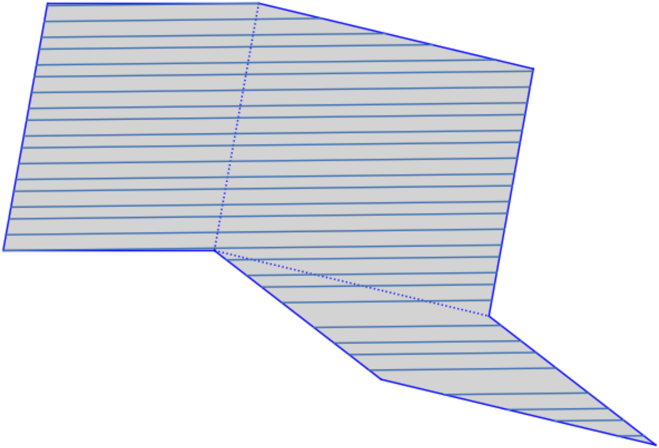

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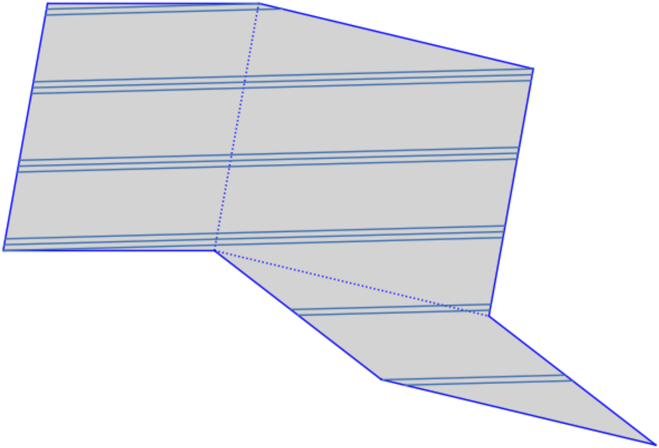
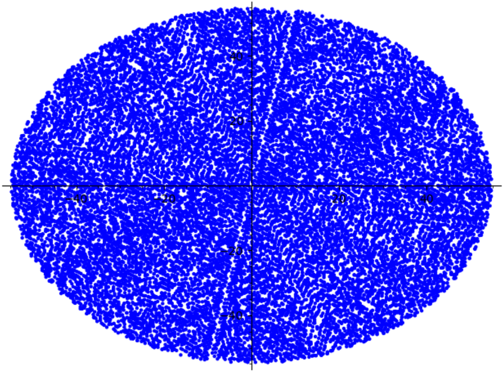


Counting Saddle Connections



Question

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- Thm. [Masur]: There are roughly L^2 saddle connections up to length L.

Asymptotics – Siegel-Veech Constants: (Not this talk)

- Let $N_{sc}((X, \omega), L)$ be the number of saddle connections on (X, ω) of length at most L.
- Thm. [Veech, Eskin-Masur] For a.e. $(X, \omega) \in \mathcal{H}(\kappa)$

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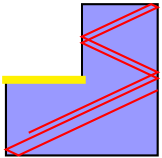
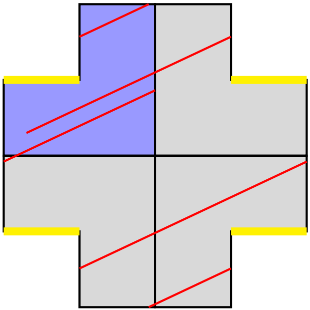
Unfolding of Billiard Table with a Missing Side



Original billiard



Unfolded billiard

The bold yellow side is a pocket or hole in the table

A Slit Translation Surface

- We pick a distinguished saddle connection β on (X, ω) and call it a *slit*
- (If there is a single singularity, every pair of saddle connections intersects at that singularity)
- We consider saddle connections that are *interiorly disjoint*
- A translation surface with such a distinguished saddle connection is called a *slit translation surface* $(X, \omega) \setminus \beta$

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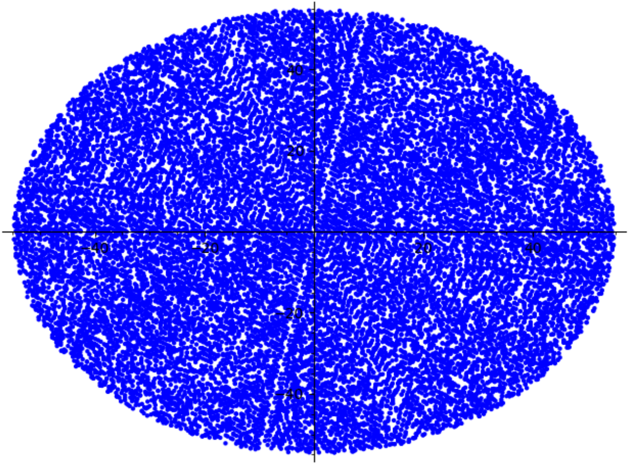
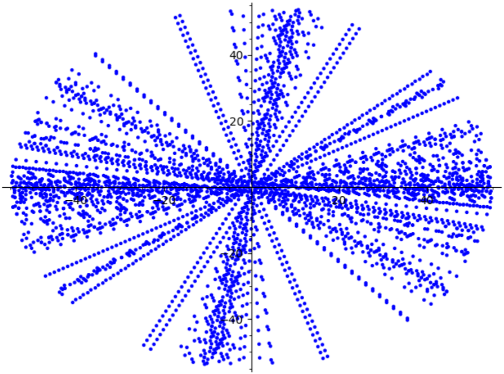
Recall the figure without the slit:



Counting Saddle Connections



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- $(X, \omega) \in \mathcal{H}_1^{hyp}(\kappa)$
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Counting Saddle Connections

Theorem (AMPS)

Let $A((X,\omega) \setminus \beta, L)$ be the set of saddle connections interiorly disjoint from β of length at most L. Then there exist c and c' depending only on the stratum such that

(1) *For every $(X,\omega) \in \mathcal{H}_1^{hyp}(\kappa)$, there exists L_0 depending on (X,ω) such that for $L \geq L_0$*

$$\#A((X,\omega) \setminus \beta, L) \leq c' \frac{L}{|\beta|} \left(\log \frac{L}{|\beta|} \right)^{d-2}.$$

(2) *For a.e. $(X,\omega) \in \mathcal{H}_1^{hyp}(\kappa)$, there exists L_0 depending on (X,ω) and β such that for $L \geq L_0$*

$$\#A((X,\omega) \setminus \beta, L) \geq c \frac{L}{|\beta|} \left(\log \frac{L}{|\beta|} \right)^{d-2}.$$

Proof Ideas

Upper Bound:

- Teichmüller renormalization: Now standard strategy.
- Make long saddle connections short and study the resulting geometry of the surface.
- If there's clustering, there must be a small area subsurface, which can be bounded.

Lower Bound:

- Horocycle renormalization: Novel technique - preserves slit direction
- Nguyen-Pan-Su Algorithm: Produces cylinders containing the slit (so the complement does not)
- Borel-Cantelli argument results in weakening to a.e. result

Thank you

Figure: https://arxiv.org/abs/2601.15993