Weak mixing behaviour for the projectivized derivative extension

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Preliminaries

Ergodic Theory: Basic recall

A **measure preserving transformation** (mpt) is a quartet $(M, \mathcal{B}, \mu, T)$ where $(M, \mathcal{B}, \mu)$ is a (probability) measure space, and

1. $T : M \to M$ is a measurable map: $E \in \mathcal{B} \implies T^{-1}(E) \in \mathcal{B}$.
2. The measure μ is T invariant: $\mu(T^{-1}(E)) = \mu(E) \; \forall \; E \in \mathcal{B}$.

A measurable set A is called invariant if $T^{-1}A = A$ mod 0.

An mpt is called **ergodic** if for any invariant set $A \in \mathcal{B}$, $\mu(A) = 0$ or 1.

An mpt is called **weak mixing** if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \left| \mu(A \cap T^{-k}B) - \mu(A)\mu(B) \right| = 0 \quad \forall A, B \in \mathcal{B}$$

For the rest of the talk we will stick to the realm of 'smooth ergodic theory', i.e. our maps will be diffeomorphisms of manifolds.

IM diffeomorphism

A diffeomorphism on a manifold M, which preserves an absolutely continuous probability measure and a measurable Riemannian metric, is said to be an **infilnitesimal measurable (IM) diffeomorphism**.

Equivalent formulation: Existence of an invariant measure for the projectivized derivative extension of the action which is absolutely continuous in the fibers.

For a detailed understanding, refer to section 3 of the work of Gunesh and Katok.

R. Gunesch, A. Katok: *Construction of weakly mixing diffeomorphisms preserving measurable Riemannian metric and smooth measure,* Discrete Contin. Dynam. Systems 6 (2000), no. 1, 61-88.

Projectivized derivative extension

Consider a smooth diffeomorphism $f : M \to M$ of a smooth manifold.

Then its **derivative extension** is the pair (f, df) considered as a transformation defined on the tangent bundle $\mathrm{T}M$.

Identifying each tangent space with $\mathbb{R}^n$ and considering its projectivization, we create the projectivized tangent bundle $\mathbb{P}\mathrm{T}M$.

With some abuse of notation, we consider the pair (f, df) to be defined on the projectivized tangent bundle $\mathbb{P}\mathrm{T}M$.

Examples of IM diffeomorphisms

First examples

Theorem (Gunesch-Katok 2000)

In any compact C^∞ manifold of dimension $m \geq 2$ admitting a nontrivial C^∞ circle action there exists a $\mathbb{Z}^k$ action for any positive integer k by weakly mixing C^∞ Liouvillean diffeomorphisms preserving a C^∞ measure and a measurable Riemannian metric.

They refer to the technique used in the proof as the "conjugation-approximation" method. It was developed by Anosov and Katok in the late 1960s.

R. Gunesch, A. Katok: *Construction of weakly mixing diffeomorphisms preserving measurable Riemannian metric and smooth measure,* Discrete Contin. Dynam. Systems 6 (2000), no. 1, 61-88.

D.V. Anosov, A.B. Katok: *New examples in smooth ergodic theory. Ergodic diffeomorphisms.(Russian)* Trudy Moskov. Mat. Obsc. 23 (1970), 3-36.

Dense in closure of conjugates of Liouville rotations

Theorem (Gunesch-Kunde 2018)

Let M be a smooth compact and connected manifold of dimension $m \geq 2$ with a non-trivial circle action $\mathcal{S} = \{S_t\}_{t\in\mathbb{R}}, S_{t+1} = S_t$. For any $\mathcal{S}$-invariant smooth volume ν the following is true: If $\alpha \in \mathbb{R}$ is Liouville, then the set of ν-preserving diffeomorphisms, that are weakly mixing and preserve a measurable Riemannian metric, is dense in $\mathcal{A}_\alpha(M) = \overline{\{h \circ S_\alpha \circ h^{-1} : f \in \mathrm{Diff}^\infty(M,\nu)\}}^{C^\infty}$.

An irrational number α is Liouville if for each n, there exist integers p and $q > 1$ such that $0 < |\alpha - p/q| < 1/q^n$. The technique used in this proof was a modified version of the conjugation-approximation method. This modified verion was developed in the work of Fayad and Saprykina.

R. Gunesch, P. Kunde: *Weakly mixing diffeomorphisms preserving a measurable Riemannian metric with prescribed Liouvillean rotation behavior,* Discrete Contin. Dyn. Syst. 38 (2018), no. 4, 1615-1655.

B. Fayad, M. Saprykina: *Weak mixing disc and annulus diffeomorphisms with arbitrary Liouville rotation number on the boundary,* Ann. Sci. École Norm. Sup. (4) 38 (2005), no. 3, 339-364.

Real-analytic examples

Theorem (Kunde 2017)

There exists a weakly mixing real-analytic diffeomorphism preserving a smooth volume and a measurable Riemannian metric.

The technique here was yet another modified version of the conjugation-approximation method. This version allowed for real-analytic constructions on the torus and was originally introduced in the work of Saprykina.

P. Kunde: *Real-analytic weak mixing diffeomorphisms preserving a measurable Riemannian metric,* Ergodic Theory Dynam. Systems 37 (2017), no. 5, 1547-1569.

M. Saprykina: *Analytic nonlinearizable uniquely ergodic diffeomorphisms on* $\mathbb{T}^2$, Ergodic Theory Dynam. Systems 23 (2003), no. 3, 935-955.

Ergodic proj. derivative extensions

Theorem (Kunde 2020)

Let M be a smooth compact and connected manifold of dimension 2 with a non-trivial circle action $\mathcal{S} = \{S_t\}_{t\in\mathbb{R}}, S_{t+1} = S_t$ preserving a smooth volume ν. Then there exists a volume-preserving diffeomorphism, whose projectivized derivative extension is ergodic with respect to a measure in the projectivization of the tangent bundle which is absolutely continuous in the fibers.

The technique here was still the conjugation-approximation method. However the 'conjugates' the modified to produce the added complexity.

P. Kunde: *Smooth zero-entropy diffeomorphisms with ergodic derivative extension,* Comment. Math. Helv. 95 (2020), no. 1, 1-25.

Weak mixing proj. derivative extensions

Theorem (B-Khurana-Kunde pp)

Let M be a smooth compact and connected manifold of dimension 2 with a non-trivial circle action $\mathcal{S}$. For any $\mathcal{S}$-invariant smooth volume ν the following is true: If $\alpha \in \mathbb{R}$ is Liouville, then the set of ν-preserving diffeomorphisms, that are weakly mixing, preserve a measurable Riemannian metric, and weakly mixing with respect to $\overline{\nu}$, is dense in $\mathcal{A}_\alpha(M) = \overline{\{h \circ S_\alpha \circ h^{-1} : f \in \mathrm{Diff}^\infty(M,\nu)\}}^{C^\infty}$.

Here $\overline{\nu}$ is an invariant measure for the projectivized derivative extension of the action which is absolutely continuous in the fibers (Recall the equivalent definition for IM diffomorphisms).

This once again uses the modified conjugation-approximation method developed by Fayad-Saprykina.
Recall that a similar result was already proved by Kunde, who showed weak mixing for the base map w.r.t. ν but not for the projectivized derivative extension. Here, the arguments were tightened.

Real-analytic weak mixing proj. derivative extensions

Theorem (B-Khurana-Kunde pp)

There exists a real analytic diffeomorphism on $\mathbb{T}^2$*, preserving a volume* ν*, whose projectivized derivative extension is weakly mixing w.r.t.* $\overline{\nu}$*.*

The crucial ideas came from the work of P. Berger enabling this improvement.

P. Berger: *Analytic pseudo-rotations,* Ann. of Math. (2).

Technique: The AbC method

History

Figure: Left: D.V. Anosov by K. Jacobs, CC BY-SA 2.0 de. Right: A. Katok, CC BY-SA 3.0.

- The *approximation by conjugation* method was introduced by D.V. Anosov and A. Katok in the late 1960s.
- Also known in literature as the "AbC" or "conjugation-approximation" or "Anosov-Katok" method.

D.V. Anosov, A.B. Katok: *New examples in smooth ergodic theory. Ergodic diffeomorphisms.(Russian)* Trudy Moskov. Mat. Obsc. 23 (1970), 3-36.

Approximation by Conjugation method: Setting

Let M be a smooth compact connected manifold of dimension $d \geq 2$ admitting a non-trivial circle action $\mathcal{S} = \{S_t\}_{t \in \mathbb{S}^1}$ preserving a smooth volume μ,

e.g. torus $\mathbb{T}^2$, annulus $\mathbb{S}^1 \times [0,1]$ or disc $\mathbb{D}^2$ with standard circle action comprising of the diffeomorphisms $S_t(\theta, r) = (\theta + t, r)$.

- We construct a sequence of measure-preserving diffeomorphisms

$$T_n = H_n \circ S_{\alpha_n} \circ H_n^{-1},$$

  where
  $\alpha_n = \frac{p_n}{q_n} \in \mathbb{Q}$ with p_n, q_n relatively prime,
  $H_n = h_1 \circ h_2 \circ ... \circ h_n$ with h_i measure-preserving diffeomorphism of M.

- We need a criterion for the aimed property expressed on the level of the maps T_n and appropriate partitions of the manifold.

Scheme

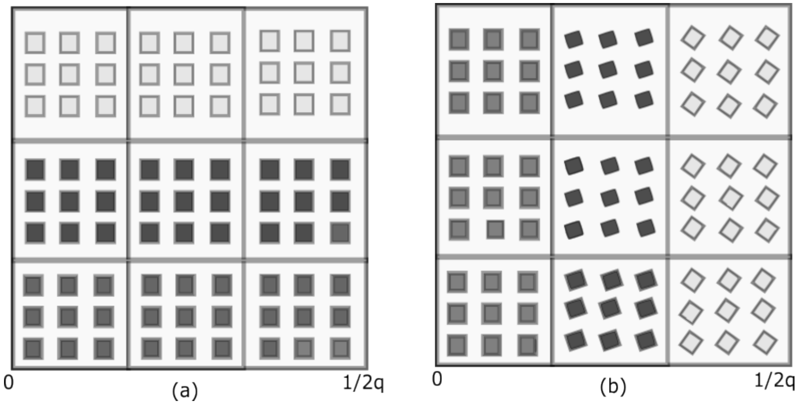
Construction of $T_n = H_n \circ S_{\alpha_n} \circ H_n^{-1}$:

- Initial step: Choose $\alpha_0 = \frac{p_0}{q_0}$ arbitrary, $T_0 = S_{\alpha_0}$.
- Step $n+1$:
  Put $\alpha_{n+1} = \frac{p_{n+1}}{q_{n+1}} = \alpha_n + \frac{1}{l_n \cdot k_n \cdot q_n^2}$ with parameters $l_n, k_n \in \mathbb{Z}$.
  The conjugation map h_{n+1} and the parameter l_n are chosen such that $h_{n+1} \circ S_{\alpha_n} = S_{\alpha_n} \circ h_{n+1}$ and T_{n+1} imitates the desired property with a certain precision.
  Then the parameter k_n is chosen large enough to guarantee closeness of T_{n+1} to T_n in the C^∞-topology:

$$\begin{aligned} T_{n+1} &= H_{n+1} \circ S_{\alpha_{n+1}} \circ H_{n+1}^{-1} \\ &= H_n \circ h_{n+1} \circ S_{\alpha_n} \circ S_{\frac{1}{l_n \cdot k_n \cdot q_n^2}} \circ h_{n+1}^{-1} \circ H_n^{-1} \\ &= H_n \circ S_{\alpha_n} \circ h_{n+1} \circ S_{\frac{1}{l_n \cdot k_n \cdot q_n^2}} \circ h_{n+1}^{-1} \circ H_n^{-1} \approx H_n \circ S_{\alpha_n} \circ H_n^{-1} = T_n \end{aligned}$$

$\Longrightarrow$ Convergence of the sequence $(T_n)_{n \in \mathbb{N}}$ to a limit diffeomorphism with the aimed properties

Criterion for IM diffeomorphisms



Real analytic constructions: Success and obstructions

- This method has been a central technique in construction of zero entropy smooth diffeomorphisms on smooth manifolds admitting a non trivial action of the circle.
- However, in construction of real-analytic diffeomorphisms, the success of this method remained limited to the torus and odd dimensional spheres.
- Early conjectures due to Birkhoff and Kolmogorov, was believed to be resolvable by this method but no proofs could be obtained. For example see the following excerpt:

> PROBLEM 7.5. Does there exist an area preserving topologically transitive real analytic diffeomorphism f of the disc $\mathbb{D}^2$ with either of the following properties:
>
> (1) the restriction of f to the boundary is an irrational rotation;
> (2) f has zero topological entropy;
> (3) f is C^2 close to the identity?

B. Fayad, A. Katok: *Constructions in elliptic dynamics,* Ergodic Theory Dynam. Systems 24 (2004), no. 5, 1477-1520.

Real analytic constructions: A recent breakthrough

In the abstract of his paper, Pierre Berger states:

"We construct analytic symplectomorphisms of the cylinder or the sphere with zero or exactly two periodic points that are not conjugate to a rotation. In the case of the cylinder, we show that these symplectomorphisms can be chosen to be ergodic or, to the contrary, with local emergence of maximal order. In particular, this disproves a conjecture of Birkhoff (1941) and solves a problem of Herman (1998). One aspects of the proof provides a new approximation theorem; it enables the implementation of the Anosov-Katok scheme in new analytic settings."

This and following works due to Pierre Berger and his student Yann Delaporte generalize the Approximation by Conjugation method to the disk and annulus and allows for more flexibility on the torus. We were able to use ideas from Berger's work to upgrade the our result to the real-analytic realm.

Note that a fully general real-analytic Approximation by Conjugation method is still not available.

Thank you!