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Fixed point approach to constrained convex minimization problems and equilibrium problems

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Outline

Abstract

In this presentation, we discuss fixed point approach to a constrained convex minimization problem and a generalized equilibrium problem. We also explore some iterative schemes in approximating a common solution of a constrained convex minimization problem, an equilibrium problem and a fixed point problem of a nonlinear mapping in the literature. Finally, we review the iterative schemes in the literature compared to the well known gradient projection algorithm (GPA).

Motivation and Introduction

Throughout this presentation H denotes a real Hilbert space with inner product $\langle .,. \rangle$ and the induced norm $\|\cdot\|$, and C denotes a nonempty closed convex subset of H. A mapping $T : C \to C$ is said to be

- $L-$Lipschitzian mapping if for some $L \geq 0$,

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \text{for all } \; x, y \in C. \tag{1}$$

- a **contraction mapping** if $0 \leq L < 1$ in (1).
- a nonexpansive mapping if $L = 1$ in (1).
- a **firmly nonexpansive** if $2T - I$ is nonexpansive, or equivalently,

$$\langle x - y, Tx - Ty \rangle \geq \|Tx - Ty\|^2 \quad \forall x, y \in C.$$

- an α-**averaged mapping** for some $\alpha \in (0, 1)$ (see Tian and Liu [26]), if there is a nonexpansive mapping $S : C \to C$ such that

$$T = (1 - \alpha)I + \alpha S.$$

Motivation and Introduction

Definition 1 ((see Xu [31], Browder and Petryshyn [7]))

A mapping $T : C \to C$ is said to be **monotone** *if*

$$\langle x - y, Tx - Ty \rangle \geq 0 \text{ for all } x, y \in C;$$

and is called ν-inverse strongly monotone (for short, ν-ism) for some $\nu > 0$, if

$$\langle x - y, Tx - Ty \rangle \geq \nu \| Tx - Ty \|^2 \text{ for all } x, y \in C.$$

The monotone operators have been widely used to solve practical problems in various fields such as optimization problems, traffic assignment problems, equilibrium problems, radiation therapy, and so on. See **Browder [6], Byrne [9], Combettes and Hirstoaga [12], Han and Lo [14], Moudafi and Théra [17], Xu [28, 29, 31], Yazdi [32, 33, 34], Yazdi and Sababe [35, 36]** and references therein.

Motivation and Introduction

Let us discuss the three key problems of our focus in this presentation.
PROBLEM 1. The fixed point problem.
If $T : C \to C$ is a function, then the fixed point problem is the problem of finding a point $x_0 \in C$ such that

$$f(x_0) = x_0 \tag{2}$$

and we denote the set of all fixed points of T by $Fix(T)$.
We recall the celebrated fixed point theorem:

1. **Brouwer Theorem**
2. **Shauder Theorem**
3. **Banach Contraction Mapping Theorem**
4. **Kirk-Gohde-Browder Theorem for nonexpansive mappings**
5. **Goebel-Kirk Theorem for asymptotically nonexpansive mappings**

PROBLEM 2. The constrained convex minimization problem
If $g : H \to \mathbb{R}$ is a convex and continuously Frechet differentiable function, then the constrained convex minimization problem is the problem of minimizing g over the constraint set C; that is,

$$\min_{x \in C} g(x), \tag{3}$$

and we denote the set of solutions of (3) by U; that is,

$$U = \{u \in C : f(u) = \min_{x \in C} g(x)\}.$$

The widely considered approximation method to solve these problems is the gradient projection algorithm(GPA). If g is (Frechet) differentiable, then the GPA generates a sequence $\{x_n\}$ via the following recursive formula:

$$\begin{cases} x_0 \in C, \\ x_{n+1} = P_C(x_n - \lambda \nabla g(x_n)) \text{ for all } n = 0, 1, 2, \cdots \end{cases} \tag{4}$$

PROBLEM 3. Equilibrium problem

Let $\phi : C \times C \to \mathbb{R}$ be a bi-function. In 1994, Blum and Oettli [3] introduced an equilibrium problem (EP) as the problem of finding $u \in C$ such that

$$\phi(u, v) \geq 0 \quad \textbf{for all} \quad v \in C. \tag{5}$$

The set of solutions of (5) is denoted by $EP(\phi)$. An equilibrium problem theory has motivated the study of problems which arise from

- **image restoration,**
- **computer tomography,**
- **radiation therapy treatment planning,**
- **economics, optimization, etc.**

The widely assumed condition on the bi-function involved is **monotonicity**. Equilibrium problems include many mathematical problems.

Motivation and Introduction

Many researchers considered a generalized equilibrium problem (GEP) of finding $z \in C$ such that

$$\phi(z, y) + \langle Az, y - z \rangle \geq 0 \text{ for all } y \in C, \tag{6}$$

where $A : C \to H$ is a monotone mapping. The set of solutions of (6) is denoted by $EP(\phi, A)$; that is,

$$EP(\phi, A) = \{z \in C : \phi(z, y) + \langle Az, y - z \rangle \geq 0 \ \ \forall y \in C\}.$$

In the case when $A \equiv 0$, GEP reduces to EP. Numerous problems in physics, variational inequalities, optimization, minimax problems, the Nash equilibrium problem in non cooperative games and economics reduce to finding a solution of the GEP (6) (See Moudafi and Thèra [17], Moudafi [18, 19], Xu [31], Yazdi [32], Yazdi and Sababe [36], and the references therein).

Motivation and Introduction

Theorem 2 (Xu [30] (Inverse Problems 26 (2010), 105018.))

Let C be a nonempty closed convex subset of a real Hilbert space H. Assume that $g : H \to \mathbb{R}$ is a convex function whose gradient ∇g is an L-Lipschitzian mapping with $L > 0$. Assume that the constrained convex minimization problem in (2) is consistent. Then

(i) ∇g *is an* $\frac{1}{L}$*-ism.*

(ii) *For* $\lambda > 0$, *the mapping* $I - \lambda \nabla g$ *is* $\frac{\lambda L}{2}$*−averaged .*

(iii) *The composite* $P_C(I - \lambda \nabla g)$ *is* $(\frac{2 + \lambda L}{4})$*-avereged for* $0 < \lambda < \frac{2}{L}$.

(iv) $x^* \in C$ *solves the minimization problem (2) if and only if for any fixed positive number* $\lambda > 0$ $x^* \in C$ *solves the fixed point equation*

$$x^* = P_C(I - \lambda \nabla g)x^*.$$

Definition 3 (Blum and Oettli [3])

*Let C be a nonempty closed convex subset of H. A bi-function $\phi : C \times C \to \mathbb{R}$ is said to satisfy "**Condition A**" if the following four conditions hold:*

(A_1) $\phi(x,x) = 0$ *for all* $x \in C$*;*

(A_2) ϕ *is monotone; that is,*

$$\phi(x,y) + \phi(y,x) \leq 0 \ \text{ for all } \ x, y \in C;$$

(A_3) *for each* $x, y, z \in C$*,*

$$\lim_{t \downarrow 0} \phi(tz + (1-t)x, y) \leq \phi(x,y);$$

(A_4) *for each* $x \in C$*,* $y \mapsto \phi(x,y)$ *is convex and weakly lower semi continuous.*

Motivation and Introduction

Theorem 4 (Combettes and Hirstoaga[12])

Assume that $\phi : C \times C \to \mathbb{R}$ *satisfies* **Condition A**. *For* $r > 0$ *define a mapping* $Q_r : H \to C$ *as follows:*

$$Q_r x = \{z \in C : \phi(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \text{ for all } y \in C\}, \quad x \in H.$$

Then, the following hold:

(i) Q_r *is single-valued;*

(ii) Q_r *is firmly nonexpansive; that is, for all* $x, y \in H$

$$\|Q_r x - Q_r y\|^2 \leq \langle Q_r x - Q_r y, x - y\rangle;$$

(iii) $F(Q_r) = EP(\phi)$*;*

(iv) $EP(\phi)$ *is closed and convex.*

Motivation and Introduction

Remark 5

It follows from Theorem 2 and Theorem 4 that both Problem 2 and Problem 3 are also fixed point problems. So we are left with the question of identifying key properties of the operators obtained by changing the problems to fixed point problems.

Advantages of having fixed point problems:

- $\surd$ Wide domain of application;
- $\surd$ Well developed theoretical proofs for existence and uniqueness of solutions;
- $\surd$ Having well developed iterative algorithms that can be coded easily;
- $\surd$ applicability in computing that requires high reliability; and so on.

In spite of this all still slow convergence rate of some fixed point algorithms is one of the main challenges.

We impose the following assumptions on the mappings and spaces involved.

(B1) Assume that $g : C \to \mathbb{R}$ is a real-valued convex function whose ∇g is a Lipschitzian mapping with Lipschitz constant $L > 0$. The solution set of the minimization problem $\min\{g(x) : x \in C\}$ is denoted by U; that is,

$$U = \{z \in C : \quad g(z) = \min_{y \in C} g(y)\}.$$

Let $\{\lambda_n\}$ be a sequence of positive real numbers in $(0, \frac{2}{L})$ such that

$$P_C(I - \lambda_n \nabla g) = (\frac{2 - \lambda_n L}{4})I + (\frac{2 + \lambda_n L}{4})T_n = \gamma_n I + (1 - \gamma_n)T_n$$

where $T_n : C \to C$ is nonexpansive, and $\gamma_n = \dfrac{2 - \lambda_n L}{4}$. Assume that $\lim_{n\to\infty} \gamma_n = 0$ (or alternatively, $\lim_{n\to\infty} \lambda_n = \frac{2}{L}$) and $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$.

(B2) Let $\phi : C \times C \to \mathbb{R}$ be a bi-function satisfying **Condition A**, $B : C \to C$ an α-ism mapping. The solution set of the generalized equilibrium problem is denoted by $EP(\phi, B)$; that is,

$$EP(\phi, B) = \{z \in C : \phi(z, y) + \langle y - z, Bz\rangle \geq 0 \quad \textit{for all } y \in C\}.$$

(B3) Let $f : C \to C$ be a contraction mapping with contraction constant $k \in [0, 1)$.

(B4) Let $S : C \to C$ be a nonlinear mapping with fixed point set $Fix(S)$.

(B5) Assume that

$$\Sigma = U \cap EP(\phi, B) \cap Fix(S) \neq \emptyset.$$

Iterative Algorithms

In 2007, Plibtieng and Punpaneng [22] introduced an iterative scheme for finding a common element of the set of solutions of (4) and the set of fixed points of a nonexpansive mapping in a Hilbert space as follows:

$$\begin{cases} \phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \quad \textit{for all } y \in H, \\ x_{n+1} = \alpha_n k f(x_n) + (I - \alpha_n A)Su_n, \quad n \geq 1, \end{cases} \tag{7}$$

where

- √ A is strongly positive bounded linear operator on H,
- √ S is a nonexpansive self-mapping of H such that $Fix(S) \cap EP(\phi) \neq \emptyset$,
- √ $\{\alpha_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty)$.

They proved that the sequence $\{x_n\}$ defined in (7) converges strongly to the unique solution of a certain variational inequality.

Iterative Algorithms

In 2020, Yazdi [33] introduced the following explicit composite iterative method for finding the common solution of a generalized equilibrium problem and a constrained convex minimization problem:

$$\begin{cases} \phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle + \langle Ax_n, y - u_n\rangle \geq 0 & \textbf{for all } y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T_n u_n, & n \geq 1, \end{cases} \tag{8}$$

where $\phi : C \times C \to \mathbb{R}$ is a bi-function, ∇g is an L-Lipschitzian mapping with $L \geq 0$ such that $U \cap EP(\phi, A) \neq \emptyset$, $f : C \to C$ is a contraction with the constant $k \in [0, 1)$ and $A : C \to C$ is an $\alpha - ism$ mapping, $x_1 \in C$, $\{\alpha_n\} \subset [0, 1]$, $\{r_n\} \subset [a, b] \subset (0, 2\alpha)$, $P_C(I - \lambda_n \nabla g) = s_n I + (1 - s_n) T_n$, $s_n = \frac{2 - \lambda_n L}{4}$ and $\{\lambda_n\} \subset (0, \frac{2}{L})$. The author proved that the sequences $\{x_n\}$ and $\{u_n\}$ generated by (8) converge strongly to $q \in U \cap EP(\phi, A)$ under certain conditions, and showed that q solves certain variational inequality.

Iterative Algorithms

In 2024, Yazdi and Sababe [36] proposed the two-layer iteration process defined as

$$\begin{cases} x_1 \in C, \\ \phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle\rangle \geq 0 & \textbf{for all } y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tu_n, & n \geq 1, \end{cases} \tag{9}$$

where

- $\surd$ $T : C \to C$ is an $\alpha-$strongly quasi-nonexpansive mapping such that $I - T$ is demiclosed at zero,
- $\surd$ $\{\alpha_n\} \subset [0, 1]$, and $\{r_n\} \subset (0, \infty)$.

The authors proved that the sequences $\{x_n\}$ and $\{u_n\}$ generated by (9) converge strongly to $q \in Fix(T) \cap EP(\phi)$ under certain conditions on the parameters.

In 2025, Sangago et al. [25] proposed the following viscosity iterative algorithm to approximate a common solution of a constrained convex minimization problem, a generalized equilbrium problem, and fixed point problem of directed nonexpansive mapping:

$$\begin{cases} x_1 \in C, \\ \phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle + \langle y - u_n, Bx_n\rangle \geq 0 \textbf{ for all } y \in C, \\ v_n = \alpha_n x_n + (1-\alpha_n)T_n u_n \\ x_{n+1} = \beta_n f(x_n) + (1-\beta_n)Sv_n, \quad n \geq 1. \end{cases} \tag{10}$$

where

- √ $S : C \to C$ is a directed nonexpansive mapping,
- √ $B : C \to C$ is an $\alpha-$ism.

The authors proved that the sequences $\{x_n\}$, $\{v_n\}$, and $\{u_n\}$ generated by (10) converge strongly to $q \in Fix(S) \cap EP(\phi, B) \cap U$ under mild conditions on the parameters.

In 2026, Ramatebele et al. introduce a mixed Mann-type viscosity approximation scheme to approximate a common solution of the fixed point problem of directed nonexpansive mapping S, the generalized equilibrium problem and the constrained convex minimization problem as follows:

$$\begin{cases} x_1 \in C, \\ \phi(z_n, y) + \frac{1}{r_n}\langle y - z_n, z_n - x_n\rangle + \langle y - z_n, Bx_n\rangle \geq 0 \textbf{ for all } y \in C, \\ w_n = \alpha_n x_n + (1-\alpha_n)P_C(z_n - \lambda_n \nabla g(z_n)) \\ x_{n+1} = \beta_n f(x_n) + (1-\beta_n)Sw_n, \quad n \geq 1. \end{cases} \tag{11}$$

and proved the following theorem.

Theorem 6 (Ramatebele, Sangago and Tufa (2026))

Let S be a directed nonexpansive mapping. Suppose $\{\alpha_n\}$, $\{\beta_n\}$ and $\{r_n\}$ are real sequences satisfying the following conditions:

$$(i)\quad \{\alpha_n\} \subset (0,1],\ \lim_{n\to\infty} \alpha_n = 0,\ \sum_{n=1}^{\infty} \alpha_n = \infty \text{ and } \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty;$$

$$(ii)\quad \{\beta_n\} \subset (0,1],\ \lim_{n\to\infty} \beta_n = 0,\ \sum_{n=1}^{\infty} \beta_n = \infty \text{ and } \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty;$$

$$(iii)\quad \{r_n\} \subset [a,b] \subset (0,2\alpha) \text{ and } \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty.$$

Then the sequences $\{x_n\}$, $\{w_n\}$ and $\{z_n\}$ defined by (11) converge strongly to $q = P_{\Sigma} f(q)$.

Concluding Remarks

The above algorithms are directly or indirectly extensions of the Gradient Projection Algorithm. The following questions remain open for research:

1. The rate of convergence of these iterative algorithms are not well investigated.
2. In comparison to the existing algorithms such as Picard, Mann or Ishikawa and their extensions, the challenge of coding these algorithms have not been well investigated.
3. In the last two results we have the new class of mappings called directed nonexpansive mappings is introduced. It was shown that this class of mappings is between the class of firmly nonexpansive and the class of nonexpansive mappings. However, complete characterization of this class of mappings is still an open problem.

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THANK YOU!