Rigidity and flexibility of entropies of boundary maps associated to Fuchsian groups

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Set-up and history

- Γ:= cocompact torsion free Fuchsian group (surface group), $\Gamma < \mathrm{PSU}(1,1)$ acting properly discontinuously by orientation-preserving isometries on the unit disk $\mathbb{D}$.

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  •1991: Adler-Flatto studied Bowen-Series maps for surface groups using fundamental $8g-4$-gons.
  • 2017: S.K.-Ugarcovici, 2019: Abrams-S.K. studied families of boundary maps $f_{\bar{A}}$ for surface groups related to fundamental $8g-4$-gons and proved that 2-dim their natural extensions $F_{\bar{A}}$ have global attractors with finite rectangular structure.

$(8g-4)$-gonal fundamental domain $\mathcal{F}$

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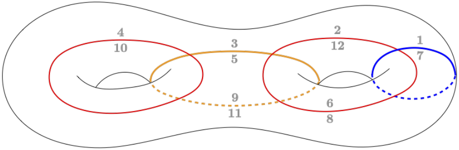

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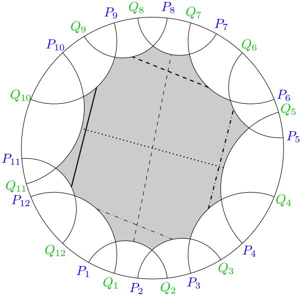

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Identification of sides by the involution σ:

$$\sigma(k) := \begin{cases} 4g-k \pmod{8g-4} & \text{if } k \text{ is odd} \\ 2-k \pmod{8g-4} & \text{if } k \text{ is even} \end{cases}$$

Dehn, Fenchel, Nielsen, Koebe, Adler-Flatto [AF91].

[AF91] R. Adler, L. Flatto. Geodesic flows, interval maps, and symbolic dynamics, *Bull. Amer. Math. Soc.* **25** (1991), No. 2, 229–334.

Marked canonical polygon

Thus obtained $8g-4$-gon $\mathcal{F}$ does not need to be regular, but satisfies the following properties:

1. The identified sides k and $\sigma(k)$ have equal length.
2. The angles at glued vertices k and $\sigma(k)+1$ add up to π

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☛ $\Gamma\backslash\mathcal{F}$ is decomposed into $2g-1$ vertices, $4g-2$ edges, and 1 (s. c.) face, and by Euler's formula has genus g. By Poincaré's Polygon Theorem $\Gamma = \langle T_k\rangle, \{T_k :$ side $k \to$ side $\sigma(k)\}$ is a Fuchsian group of the first kind with signature $(g;-;0)$, i.e. a surface group.

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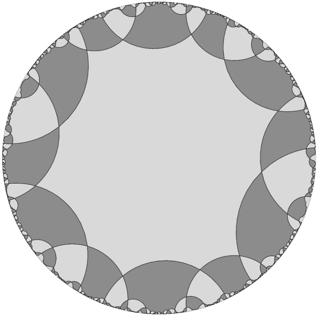

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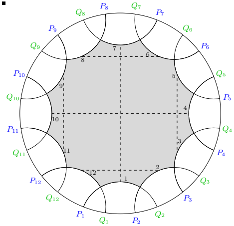

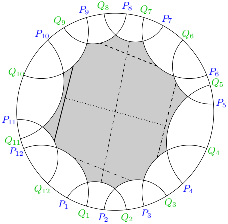

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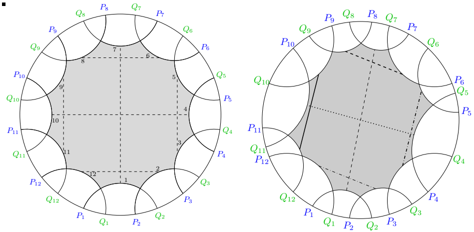

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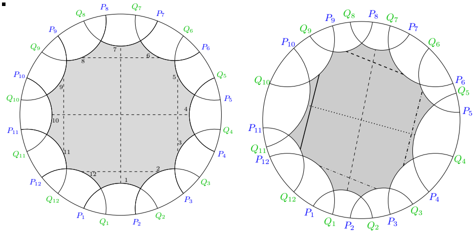

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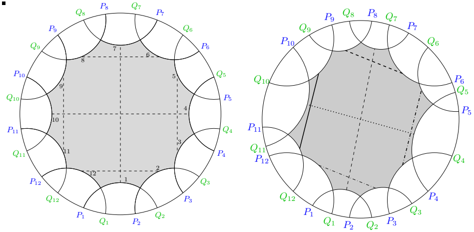

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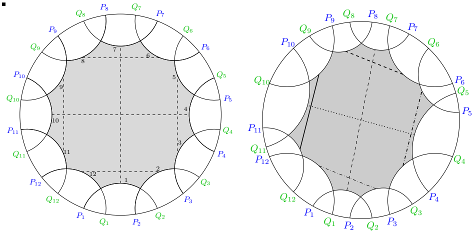

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☛ This construction gives a less common representation of the Teichmüller space $\mathcal{T}(g)$ for $g \geq 2$: it is the space of marked canonical $(8g-4)$-gons in $\mathbb{D}$ up to an isometry of $\mathbb{D}$.

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• For any marked canonical fundamental $8g-4$-gon $\mathcal{F} \in \mathcal{T}(g)$, let
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• For any multi-parameter $\bar{A} = \{A_1, \ldots, A_{8g-4}\}, \quad A_k \in [P_k, Q_k]$
we define the boundary map $f_{\bar{A}} : \mathbb{S} \to \mathbb{S}$

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$f_{\bar{P}}$ was referred to in [AF91] as the Bowen–Series boundary map, (although Bowen-Series construction [BS79] used $4g$-sided polygons).

[BS79] R. Bowen, C. Series. Markov maps associated with Fuchsian groups, *Inst. Hautes Études Sci. Publ. Math.* No. 50 (1979), 153–170.
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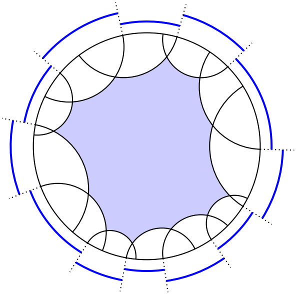
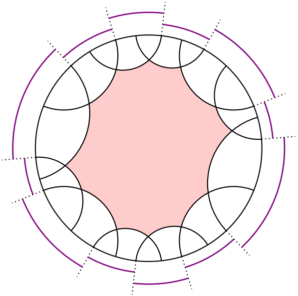
Deformations



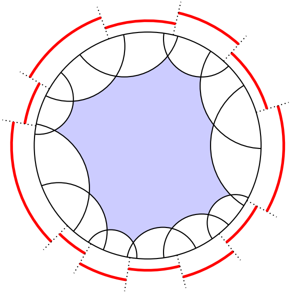
Original ($\bar{P}$)

Teichmüller

Multi-parameter



Change polygon



Change interval where each T_k is used

Multi-parameters

• $\bar{A} = \{A_1, \ldots, A_{8g-4}\}$ is extremal if $\forall k$ either $A_k = P_k$ or $A_k = Q_k$;
e.g. $\bar{P}$ and $\bar{Q}$ are extremal.
• [KU17]: If $A_k \in (P_k, Q_k)$ $\bar{A}$ satisfies the cycle property, i.e. $\exists$ positive integers m_k, n_k such that $f^{m_k}(T_k A_k) = f^{n_k}(T_{k-1} A_k)$

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- If $\bar{A}$ is extremal or satisfies the short cycle property, then $f_{\bar{A}}$ admits a unique smooth invariant ergodic measure $\mu_{\bar{A}}$ related to the Liouville measure for the geodesic flow.

[KU17] S. Katok, I. Ugarcovici. Structure of attractors for boundary maps associated to Fuchsian groups. *Geometriae Dedicata* **191** (2017), 171–198.

Rigidity and Flexibility theorems

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Let $S = \Gamma \backslash \mathbb{D}$ be a surface of genus $g \geq 2$ and let $\mathcal{F} \in \mathcal{T}(g)$.

Theorem 1. [AKU-2024] Rigidity of topological entropy.

$\forall \bar{A} = \{A_1, \ldots, A_{8g-4}\}$ with $A_k \in [P_k, Q_k]$, the map $f_{\bar{A}} : \mathbb{S} \to \mathbb{S}$ has the same topological entropy $h_{\text{top}}(f_{\bar{A}}) = \log(4g - 3 + \sqrt{(4g-3)^2 - 1})$.
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Theorem 2. [AKU-2022] Flexibility of measure-theoretic entropy

(in the spirit of flexibility program suggested by Anatole Katok). Let $\bar{A}$ be extremal or satisfy the short cycle property (an open set of partitions). Then

❶ For each $\mathcal{F} \in \mathcal{T}(g)$, $h_{\mu_{\bar{A}}}(f_{\bar{A}}) = \frac{\pi^2(4g-4)}{\text{Perimeter}(\mathcal{F})} = \pi \cdot \frac{\text{Area}(\mathcal{F})}{\text{Perimeter}(\mathcal{F})}$.

❷ The maximum value of $h_{\mu_{\bar{A}}}(f_{\bar{A}})$, $H(g) = \frac{\pi^2(4g-4)}{(8g-4)\,\text{arccosh}(1+2\cos\frac{\pi}{4g-2})}$ is achieved on the surface with $\mathcal{F}_{\text{reg}}$.

❸ For each $t \in (0, H(g)]$ $\exists$ $\mathcal{F} \in \mathcal{T}(g)$ such that $h_{\mu_{\bar{A}}}(f_{\bar{A}}) = t$.

[AKU-2024] A.Abrams, S.Katok, I,Ugarcovici, In *A Vision for Dynamics - The Legacy of Anatole Katok* , 19 - 47, Cambridge University Press, 2024

[AKU-2022] A.Abrams, S.Katok, I,Ugarcovici, *Ergod. Th. & Dynam Sys.* **42** (2022), 389 - 401

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- Alseda-Misiurewicz (2015): piecewise continuous non-Markov

Steps of the proof of Theorem 1

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① Topological entropy for Markov cases: extremal $\bar{A}$. Since $f_{\bar{A}}(P_k), f_{\bar{A}}(Q_k) \in \bar{P} \cup \bar{Q}$, partition of $\mathbb{S}$ into intervals $I_1, \dots, I_{16g-8}$:

$$I_{2k-1} := [P_k, Q_k], \qquad I_{2k} := [Q_k, P_{k+1}], \qquad k = 1, \dots, 8g-4,$$

for any extremal $\bar{A}$, $f_{\bar{A}}$ is Markov with $M_{\bar{A}} = (m_{i,j})$ over the alphabet $\{1, \dots 16g-8\}$, $m_{i,j} := \begin{cases} 1 & \text{if } f_{\bar{A}}(I_i) \supset I_j \\ 0 & \text{otherwise} \end{cases}$.

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$$\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&1&1&1&1&1&1&1\\
0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0\\
1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&1&1&1&1&1\\
0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&1&1&1\\
0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1
\end{pmatrix}
\quad
\begin{pmatrix}
1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0\\
1&1&1&1&1&1&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0\\
1&1&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1\\
1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0\\
1&1&1&1&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1
\end{pmatrix}$$

Steps of the proof of Theorem 1

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$$\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&1&1&1&1&1&1&1\\
0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0\\
1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&1&1&1&1&1\\
0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&1&1&1\\
0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1
\end{pmatrix}
\quad
\begin{pmatrix}
1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0\\
1&1&1&1&1&1&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0\\
1&1&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1\\
1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0\\
1&1&1&1&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1
\end{pmatrix}$$

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$$\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&1&1&1&1&1&1&1\\
0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0\\
1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&1&1&1&1&1\\
0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&1&1&1\\
0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1
\end{pmatrix}
\qquad
\begin{pmatrix}
1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0\\
1&1&1&1&1&1&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0\\
1&1&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1\\
1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0\\
1&1&1&1&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1
\end{pmatrix}$$

- The number of $\bar{A}$-admissible words of length n grows $\approx$ as λ^n, where $\lambda = 4g - 3 + \sqrt{(4g-3)^2 - 1}$, $\Rightarrow$ the maximal eigenvalue of every $M_{\bar{A}}$ is λ.
- Thus if $\bar{A}$ is extremal, $h_{\text{top}}(f_{\bar{A}}) = \log \lambda$, so $h_{\text{top}}(f_{\bar{P}}) = h_{\text{top}}(f_{\bar{Q}}) = \log \lambda$.

② Conjugacy to a constant slope map: $\ell_{\bar{A}} = \psi_{\bar{A}} \circ f_{\bar{A}} \circ \psi_{\bar{A}}^{-1}$.

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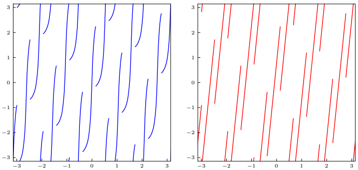

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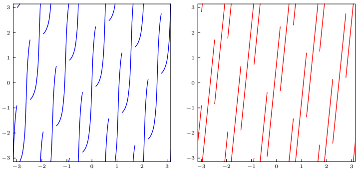

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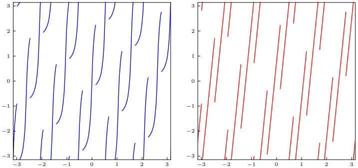

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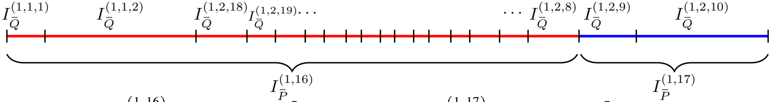

For $g = 2$, $I_{\bar{P}}^{(1,16)}$ as a union of 17 $\bar{Q}$-cylinders (left) and $I_{\bar{P}}^{(1,17)}$ as a union of 2 $\bar{Q}$-cylinders (right)

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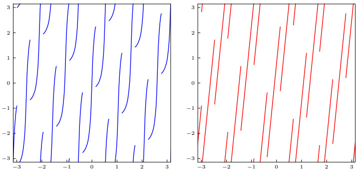

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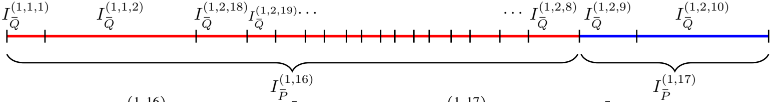

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③ Completion of the proof: the conjugacy $\psi_{\bar{P}}$ works for all $\bar{A} = \{A_1, \dots, A_{8g-4}\}$ with $A_k \in [P_k, Q_k]$. $\psi_{\bar{P}} \circ f_{\bar{A}} \circ \psi_{\bar{P}}^{-1}$ is piecewise linear with the same slope λ, hence by [M-S 1980] $h_{\text{top}}(f_{\bar{A}}) = \log \lambda$.

Steps of the proof of Theorem 2

① The smooth invariant measure.

In coordinates (u, w, s), where s is the arclength along geodesic from u to w, the Liouville volume on $T^1\mathbb{D}$ can be written as $\mathrm{d}\omega = 2\mathrm{d}m$, where $\mathrm{d}m = \mathrm{d}\nu \mathrm{d}s$, $\mathrm{d}\nu = \dfrac{|\mathrm{d}u|\,|\mathrm{d}w|}{|u-w|^2}$ is the smooth measure on the space of oriented geodesics $\mathbb{D}$ modeled as $\mathbb{S} \times \mathbb{S} \setminus \Delta$, called geodesic current.

☛ We defined a 2-dim map $F_{\bar{A}} : \mathbb{S} \times \mathbb{S} \setminus \Delta \to \mathbb{S} \times \mathbb{S} \setminus \Delta$ by

$$F_{\bar{A}}(u, w) = (T_k u, T_k w) \qquad \text{if } w \in [A_k, A_{k+1}).$$

[KU17]: $F_{\bar{A}}$ has a global attractor $\Omega_{\bar{A}}$ with finite rectangular structure.

☛ The restriction of $F_{\bar{A}}$ to $\Omega_{\bar{A}}$ is the natural extension map of $f_{\bar{A}}$.

☛ $F_{\bar{A}} : \Omega_{\bar{A}} \to \Omega_{\bar{A}}$ preserves the smooth probability measure $d\nu_{\bar{A}} = \frac{d\nu}{\int_{\Omega_{\bar{A}}} d\nu}$.

☛ The geodesic flow on S φ^t is a special flow over a cross-section parametrized by $\Omega_{\bar{A}}$, s.t. the first return map to this cross-section is $F_{\bar{A}} : \Omega_{\bar{A}} \to \Omega_{\bar{A}}$ [AK19].

☛ $f_{\bar{A}}$ is a factor of $F_{\bar{A}}$ (projecting on the second coordinate), so one can obtain its smooth invariant probability measure $\mu_{\bar{A}}$ on $\mathbb{S}$ as a projection.

[KU17] S. Katok, I. Ugarcovici. Structure of attractors for boundary maps associated to Fuchsian groups. *Geometriae Dedicata* **191** (2017), 171–198.

② Formula for the entropy.

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☛ Using that the entropy of φ^t with respect to the normalized Liouville measure is equal to 1, Abramov's formula and Ambrose-Kakutani Theorem we obtained in [AK19] the formula

$$h_{\mu_{\dot{A}}}(f_{\bar{A}}) = h_{\mu_{\dot{A}}}(F_{\bar{A}}) = \frac{\pi^2(4g-4)}{\int_{\Omega_{\dot{A}}} \mathrm{d}\nu} = \pi \cdot \frac{\mathrm{Area}(\mathcal{F})}{\int_{\Omega_{\dot{A}}} \mathrm{d}\nu}.$$

(The last equality uses Gauss–Bonnet formula: $\mathrm{Area}(\mathcal{F}) = 2\pi(2g-2)$.)

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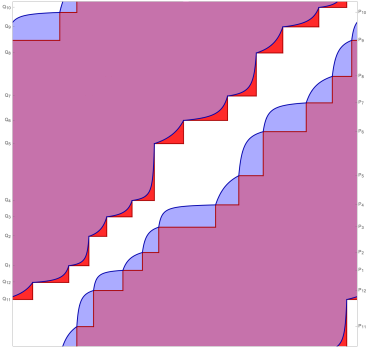
☛ A new ingredient: $\int_{\Omega_{\bar{A}}} \mathrm{d}\nu = \mathrm{Perimeter}(\mathcal{F})$. We use another map, also introduced by Adler–Flatto in [AF91], called in [AK19] geometric map. It is defined on

$$\Omega_G := \{ (u,w) : uw \text{ intersects } \mathcal{F} \} \subset \mathbb{S} \times \mathbb{S} \setminus \Delta \text{ by}$$

$$F_G(u,w) = (T_k u, T_k w) \quad \text{if } uw \text{ exits } \mathcal{F} \text{ through side } k.$$

[AK19] A. Abrams, S. Katok. Adler and Flatto revisited: cross-sections for geodesic flow on compact surfaces of constant negative curvature. *Studia Mathematica* **246** (2019), 167–202.

Theorem [AK19, [A20]: $\exists$ is a bijection map $\Phi : \Omega_G \to \Omega_{\bar{A}}$, piecewise-Möbius which maps bulges into corners.



Arithmetic set $\Omega_{\bar{P}}$ in red-and-purple, geometric set Ω_G in blue-and-purple (for an irregular polygon with $g = 2$)

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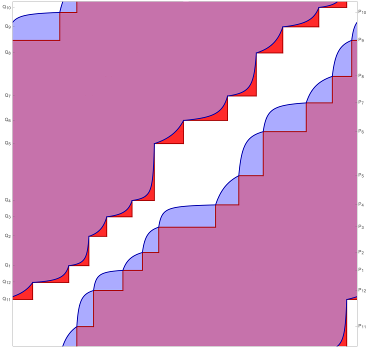

Arithmetic set $\Omega_{\bar{P}}$ in red-and-purple, geometric set Ω_G in blue-and-purple (for an irregular polygon with $g = 2$)

Corollary: $\int_{\Omega_{\bar{A}}} \mathrm{d}\nu = \int_{\Omega_G} \mathrm{d}\nu$.

[A20] A. Abrams. Extremal parameters and their duals for boundary maps associated to Fuchsian groups, *Illinois J. Math.* **65** (2021) No. 1, 153–179.

Lemma [B88, Appendix A3]: $\forall$ oriented geodesic segment s on $\mathbb{D}$,

$$\int_{\Psi^+(s)} \mathrm{d}\nu = \mathrm{length}(s),$$

where $\Psi^+(s)$ is the set of oriented geodesics intersecting s with the oriented angle at the intersection between 0 and π.

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Since $\Omega_G = \bigcup_{i=k}^{8g-4} \mathcal{G}_k$, where

$$\mathcal{G}_k = \{(u,w) \,|\, uw \text{ exits } \mathcal{F} \text{ through side } k\} = \Psi_+(\text{side } k).$$

$$\int_{\Omega_G} \mathrm{d}\nu = \sum_{k=1}^{8g-4} \int_{\mathcal{G}_i} \mathrm{d}\nu = \sum_{k=1}^{8g-4} \mathrm{length}(\text{side } k) = \mathrm{Perimeter}(\mathcal{F}).$$

Thus $h_{\mu_{\bar{A}}}(f_{\bar{A}}) = \pi \cdot \frac{\mathrm{Area}(\mathcal{F})}{\mathrm{Perimeter}(\mathcal{F})}$.

[B88] F. Bonahon. The geometry of Teichmüller space via geodesic currents, *Inventiones Mathematicae* **92** (1988), 139–162.

③ Maximum of the entropy.

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Fenchel–Nielsen coordinates on $\mathcal{T}(g) \approx \mathbb{R}^{6g-6}$ use a decomposition of S into $2g-2$ pairs of pants by $3g-3$ non-intersecting closed geodesics whose lengths can be manipulated independently.

One of them can be taken to correspond to one entire side of $\mathcal{F}$ (blue) and can be made arbitrary long, i.e. $h_{\mu_{\bar{P}}}(f_{\bar{P}})$ can be made arbitrarily small.
Since perimeter of $\mathcal{F}$ varies continuously within $\mathcal{T}(g)$, $h_{\mu_{\bar{A}}}(f_{\bar{A}})$ must take on all values between 0 and its maximum $H(g)$.

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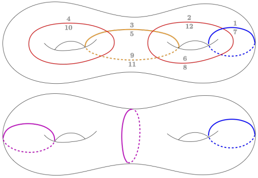


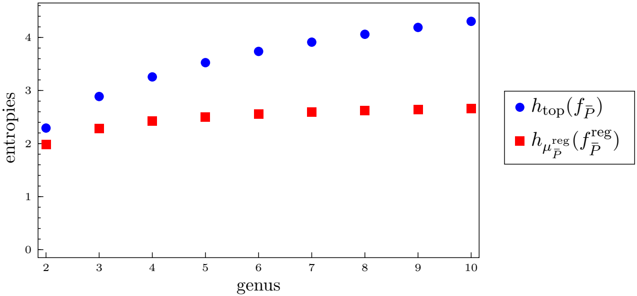
Chain of $2g$ geodesics on S forming the sides of $\mathcal{F}$ (top) and decomposition of S into $2g-2$ pairs of pants by $3g-3$ non-intersecting geodesics (bottom) for $g = 2$

[Ku] Hsu-Tung Ku, Mei-Chin Ku, Xin-Min Zhang. Isoperimetric inequalities on surfaces of constant curvature, *Canadian Journal of Mathematics* **49** (1997), 1162–1187.

Comparison of entropies

Theorem: Smooth measure is not measure of maximal entropy.

If $\bar{A}$ is extremal or has short cycles, the maximum of measure-theoretic entropy of $f_{\bar{A}}$ with respect to its smooth invariant measure $\mu_{\bar{A}}$, $h_{\mu_{\bar{A}}^{\text{reg}}}(f_{\bar{A}}^{\text{reg}}) = \frac{\pi^2(4g-4)}{(8g-4)\operatorname{arccosh}(1+2\cos\frac{\pi}{4g-2})}$ is strictly less than the topological entropy of $f_{\bar{A}} = h_{\text{top}}(f_{\bar{A}}) = \log\left(4g-3+\sqrt{(4g-3)^2-1}\right)$ for all $g \geq 2$.



Topological entropy and maximum measure-theoretic entropy for different genera

Conjectures and question

1. For any $\mathcal{F} \in \mathcal{T}(g)$ and any $\bar{A}$ with $A_k \in [P_k, Q_k]$, there exists a smooth $f_{\bar{A}}$-invariant ergodic probability measure $\mu_{\bar{A}}$.

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2. For any $\bar{A}$ with $A_k \in [P_k, Q_k]$, there exists a set $\Omega_{\bar{A}} \subset \mathbb{S} \times \mathbb{S} \setminus \Delta$ with finite rectangular structure that is a domain of bijectivity for $F_{\bar{A}}$ and moreover the global attractor of $F_{\bar{A}}$.
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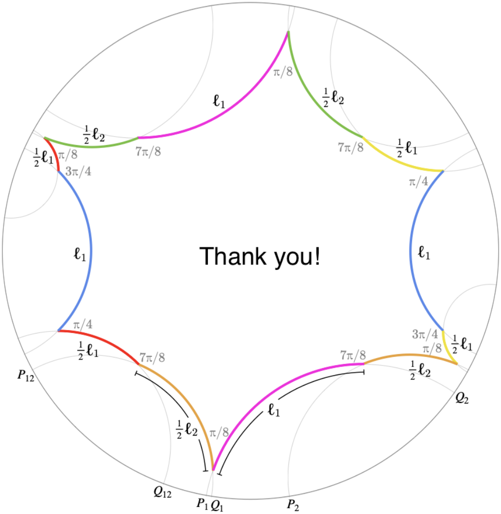
3. Is it true that for any $\bar{A}$ with $A_k \in [P_k, Q_k]$, the map $F_{\bar{A}}\big|_{\Omega_{\bar{A}}}$ is conjugate to F_{geo} by a map $\Phi_{\bar{A}} : \Omega_{\text{geo}} \to \Omega_{\bar{A}}$ that acts piecewise by Möbius transformations?


Thank you!
ℓ1
π/8
½ℓ2
7π/8
½ℓ1
π/4
3π/4
P12
Q12
P1
Q1
P2
Q2