

## Hit counting function

Let  $\Omega$  be a metric space,  $T : \Omega \rightarrow \Omega$  and  $\mu$  a  $T$ -invariant probability measure.

For  $U \subset \Omega$  ( $\mu(U) > 0$ ) define the counting function

$$Z_U^N = \sum_{j=0}^{N-1} \chi_U \circ T^j$$

For a nullset  $\Gamma \subset \Omega$  we look at

$$F(\rho, t) = \mu(Z_{B_\rho(\Gamma)}^{N_\rho(t)} = k)$$

for suitable times  $N_\rho$  where  $t$  is a parameter. We expect that as  $\rho \rightarrow 0$  one gets a limit

$$F(\rho, t) \rightarrow F(t).$$

For meaningful results one needs good mixing properties:

(i) Correlations decay at some polynomial rate

(ii)  $\mu$  is  $\phi$ -mixing w.r.t. a partition  $\mathcal{A}$  of  $\Omega$ . For right  $\phi$ -mixing that is

$$|\mu(B \cap T^{-n-j}C) - \mu(B)\mu(C)| \leq \phi(j)\mu(C)$$

for all  $B \in \mathcal{A}^n = \bigvee_{i=0}^{n-1} \mathcal{A}^i$ ,  $C \in \sigma(\bigcup_{\ell} \mathcal{A}^\ell)$  and all  $n$  and  $j$  where  $\phi(j) \rightarrow 0$  at some rate as  $j \rightarrow \infty$ .

( $\mu$  is left  $\phi$ -mixing if the RHS has the factor  $\mu(B)$  instead of  $\mu(C)$ .)

## Classical setting: A single point

$\Gamma = \{x\}$  a single point.

If  $\mu$  is  $\phi$ -mixing for a partition  $\mathcal{A}$ , then

$$Z_{A_n(x)}^{N_n} \rightarrow \text{Poisson}(t)$$

in distribution if  $x$  non-periodic and

$$N_n = \frac{t}{\mu(A_n(x))}$$

This is the Kac scaling for  $N$ .

If  $x$  is a period point with minimal period  $p$ , then  $\vartheta = \lim_{n \rightarrow \infty} \frac{\mu(A_{n+p}(x))}{\mu(A_n(x))}$  and we get

$$Z_{A_n(x)}^{N_n} \rightarrow \text{Pólya-Aeppli}(t, \vartheta)$$

in distribution, where here

$$N_n = \frac{t}{(1 - \vartheta)\mu(A_n(x))}.$$

The Pólya-Aeppli distribution is a Poisson distribution which is compounded with a geometric distribution. Its probability mass function is given by

$$\mathbb{P}(W = k) = e^{-t} \sum_{j=1}^k \vartheta^{k-j} (1 - \vartheta)^j \frac{t}{j!} \binom{k-1}{j-1}.$$

If  $(T, \Omega)$  is an Axiom A system and can be modelled by a subshift of finite time and if  $\mu$  is an equilibrium state for a potential  $f$ , then we get the Pitskel value

$$\vartheta = \vartheta(x) = e^{f^P(x) - pP(f)},$$

where  $f^P = f + f \circ T + f \circ T^2 + \dots + f \circ T^{P-1}$  and  $P(f)$  is the pressure of  $f$ .

Compound Poisson distributions are determined by a random variable  $P$  which is Poisson( $s$ ) distributed and i.i.d. r.v.s  $Y_j$ ,  $j = 1, 2, \dots$ , Then

$$W = \sum_{j=1}^P Y_j$$

is compound Poisson distributed with parameter  $t$  and

$$\lambda_\ell = \mathbb{P}(Y_j = \ell), \ell = 1, 2, \dots$$

If the  $Y_j$  are geometrical, i.e.  $\lambda_\ell = (1 - \vartheta)\vartheta^{\ell-1}$ ,  $\ell = 1, 2, \dots$  then  $W$  is Pólya-Aeppli.

The general case is when  $\Gamma \subset \Omega$  is a nullset. In the metric setting put  $\Gamma_\rho = B_\rho(\Gamma)$ ,  $\rho > 0$  for the approximating neighbourhoods.

Then put

$$\lambda_\ell(L, \rho) = \frac{\mathbb{P}(Z_{\Gamma_\rho}^L = \ell)}{\mathbb{P}(Z_{\Gamma_\rho}^L \geq 1)}$$

and

$$\lambda_\ell(L) = \lim_{\rho \rightarrow 0} \lambda_\ell(L, \rho).$$

The (double) limit

$$\lambda_\ell = \lim_{L \rightarrow \infty} \lambda_\ell(L)$$

measures the clustering at  $\Gamma$ .

## Relation to extremal indices

Put

$$\alpha_\ell(L, \rho) = \mathbb{P}(Z_{\Gamma_\rho}^L = \ell | \Gamma_\rho)$$

and  $\alpha_\ell(L) = \lim_{\rho \rightarrow 0} \alpha_\ell(L, \rho)$  and  $\alpha_\ell = \lim_{L \rightarrow \infty} \alpha_\ell(L)$  (extremal indices) then:

### Lemma

If we put  $\tau_\rho(x) = \inf\{j \geq 1 : T^j(x) \in \Gamma_\rho\}$  for the entry time to  $\Gamma_\rho$ , then

$$\mathbb{P}(\tau_\rho < L) = \mu(\Gamma_\rho) \sum_{j=1}^L \alpha_1(j, \rho).$$

In particular we get

$$\lim_{L \rightarrow \infty} \lim_{\rho \rightarrow 0} \frac{\mathbb{P}(\tau_\rho < L)}{L\mu(\Gamma_\rho)} = \alpha_1.$$

## Proof.

One has

$$\begin{aligned}\mathbb{P}(\tau_\rho < L) &= \mathbb{P}(Z_\rho^L \geq 1) \\ &= \sum_{j=0}^{L-1} \mathbb{P}(Z_\rho^j \geq 1, T^{-j}\Gamma_\rho, \tau_\rho \circ T^j \geq L - j) \\ &= \sum_{j=0}^{L-1} \mathbb{P}(\Gamma_\rho, \tau_\rho \geq L - j) \\ &= \mu(\Gamma_\rho) \sum_{k=1}^L \alpha_1(k, \rho).\end{aligned}$$

□

This lemma works with any set  $U$ ,  $\mu(U) > 0$ , in place of  $\Gamma_\rho$ .

If  $\alpha_1 > 0$  then  $\mathbb{P}(Z_\rho^L \geq 1)$  is proportional to  $L\mu(\Gamma_\rho)$  which implies that the scaling involved in  $N$  is up to a factor (namely  $\alpha_1$ ) given by the standard Kac scaling which come from the fact that

$$\int_{\Gamma_\rho} \tau_\rho d\mu = 1.$$

We then get the relation between the cluster probabilities  $\lambda_\ell$  and the extremal indices  $\alpha_\ell$ :

### Lemma

If  $\alpha_1 > 0$  and  $\sum_\ell \ell^2 \alpha_\ell < \infty$ , then

$$\lambda_\ell = \frac{\alpha_\ell - \alpha_{\ell+1}}{\alpha_1}.$$

# Assumptions

Geometric conditions:

- ▶  $T$  is expanding.
- ▶  $\mathcal{I}_n$  inverse branches of  $T^n$ .
- ▶  $\Omega \subset B_R(y_k)$  for some finitely many points  $y_k$  and  $\mu(B_R(y_k)) > \text{const } \forall k$ ,
- ▶  $n$ -cylinders are the sets  $\zeta = \varphi(B_R(y_k))$  for  $\varphi \in \mathcal{I}_n$ ,  $k$ .

Dynamic conditions:

- ▶ Decay of correlations:

$$\left| \int_{\Omega} G(H \circ T^K) d\mu - \mu(G)\mu(H) \right| \leq \mathfrak{C}(k) \|G\|_{Lip} \|H\|_{\infty}$$

- ▶ Distortion: On  $n$ -cylinders  $\zeta$ ,  $\frac{J_n(x)}{J_n(y)} \leq \mathfrak{D}(n)$ ,  $x, y \in \zeta$ ,  $\mathfrak{D}$  bounded or slowly growing.
- ▶ Contraction:  $\text{diam}\zeta < \delta(n)$  for  $n$ -cylinders  $\zeta$  and  $\delta(n) \rightarrow 0$  at some rate.
- ▶ Dimension:  $\rho^{d_1} \lesssim \mu(\Gamma_{\rho}) \lesssim \rho^{d_0}$  for some  $0 < d_0 \leq d_1 < \infty$ .
- ▶ Annulus condition:

$$\frac{\mu(\Gamma_{\rho+r} \setminus \Gamma_{\rho-r})}{\mu(\Gamma_{\rho})} = \mathcal{O}(r^{\xi}/\rho^{\beta})$$

( $r \ll \rho$ ) for some  $0 < \beta \leq \xi$ .

## Theorem

*Under the assumptions above  $Z_\rho^{N_\rho}$  converges in distribution to the compound Poisson distribution with parameters  $t$  and  $\lambda_\ell$ ,  $\ell \in \mathbb{N}$ .*

*That is*

$$\lim_{L \rightarrow \infty} \lim_{\rho \rightarrow 0} \mathbb{P}(Z_\rho^{N(L, \rho)} = k) = \nu(\{k\}),$$

*where  $\nu$  is the CP distribution on  $\mathbb{N}_0$  with the parameters  $t, \lambda_\ell$ .*

*Moreover the observation time is given by*

$$N(L, \rho) = \frac{tL}{\mathbb{P}(Z_\rho^L \geq 1)}$$

*with  $t > 0$  a parameter.*

## Kac scaling

According to the previous lemma  $\mathbb{P}(Z_\rho^L \geq 1) \sim \alpha_1 L \mu(\Gamma_\rho)$ .

If  $\alpha_1 > 0$  then the scaling in the theorem can be replaced by the Kac scaling

$$N_\rho = \frac{t}{\alpha_1 \mu(\Gamma_\rho)}$$

and the parameters  $\lambda_\ell$  can be determined from the  $\alpha_\ell$ .

# Examples

Interval maps:

For a  $C^2$  interval map  $T$  with absolutely invariant probability measure  $\mu$ ,  $\Gamma = \{x\}$  a single point.

If  $x$  is periodic with minimal period  $p$  then we get the Pitskel value

$$\vartheta = \frac{1}{DT^p(x)}$$

and  $\alpha_\ell = \lambda_\ell = (1 - \vartheta)\vartheta^{\ell-1}$ . The limiting distribution is Pólya-Aeppli.

Let  $\Gamma = \{x_1, x_2, x_3, \dots, x_n\}$ , where the  $x_i$  are periodic points with minimal periods  $p_i$ . If  $\mu$  is the a.c.i.p.m. with density  $h$ , then

$$W_\rho = \sum_{j=0}^{N-1} \chi_{\Gamma_\rho} \circ T^j \longrightarrow CP(t, \lambda_\ell)$$

in distribution as  $\rho \rightarrow 0$ , where  $N = \frac{t}{\mu(\Gamma_\rho)}$  (Kac scaling) and the clustering parameters are

$$\lambda_\ell = \frac{\sum_{i=1}^n (1 - \vartheta_i)^2 \vartheta_i^{\ell-1} h(x_i)}{\sum_{i=1}^n (1 - \vartheta_i) h(x_i)},$$

where  $\vartheta_i = |DT^{p_i}(x_i)|^{-1}$  is the Pitskel value at  $x_i$ . The limiting distribution is not Pólya-Aeppli.

Product of  $C^2$  interval maps:  $T_1, T_2$  expanding  $C^2$  interval maps, then  $T = T_1 \times T_2 : \mathbb{T} \rightarrow \mathbb{T}$  has a.c.i.p.m.  $\mu = \mu_1 \times \mu_2$  with  $\mu_i$  the a.c.i.p.m. for  $T_i$ . Let  $x$  be periodic point of  $T_1$  of minimal period  $p$  and  $\hat{\tau}_U^1(y) = \inf\{j \geq 1 : T_2^{jp} y \in U\}$  the first entry time to  $U \in [0, 1)$  for  $T_2^p$ . Similarly for the  $k$ th entry time  $\hat{\tau}_U^k$ . Put

$$\gamma_k(i) = \mathbb{P}_{\mu_2}(\hat{\tau}_{[a,b]}^k = i | [a, b])$$

for some  $0 \leq a < b \leq 1$ .

If  $\Gamma = \{x\} \times [a, b]$  then

$$W_\rho = \sum_{j=0}^{N-1} \chi_{\Gamma_\rho} \circ T^j \longrightarrow CP(t, \lambda_\ell)$$

in distribution as  $\rho \rightarrow 0$  where  $N = t/\mu(\Gamma_\rho)$ ,  $t > 0$  (Kac scaling)

and

$$\lambda_\ell = \frac{1}{\alpha_1} \sum_{i=\ell}^{\infty} \vartheta^i (\gamma_\ell(i) - \gamma_{\ell+1}(i) + \gamma_{\ell+2}(i)),$$

where  $\alpha_1 = \sum_{i=1}^{\infty} \vartheta^i (\gamma_1(i) - \gamma_2(i))$ ,  $\vartheta = |DT_1^p(x)|^{-1}$ .

Parabolic interval map:

For the Manneville-Pompeau map  $T : [0, 1) \rightarrow [0, 1)$  given by

$$T(x) = \begin{cases} x + 2^\gamma x^{1+\gamma} & x \in [0, \frac{1}{2}) \\ 2x - 1 & x \in [\frac{1}{2}, 1) \end{cases} \text{ for some parameter } \gamma \in (0, 1)$$

the origin  $x = 0$  is an indifferent fixed point with intermittent behaviour where the density  $h$  of the a.c.i.p.m. is singular ( $h(x) \sim x^{-\gamma}$  near 0). If  $\Gamma = \{0\}$  then  $\alpha_1 = 0$  and the Kac scaling cannot be used.

In this case

$$W_\rho = \sum_{j=0}^{N-1} \chi_{B_\rho(0)} \circ T^j \longrightarrow \text{Poisson}(t)$$

in distribution as  $\rho \rightarrow 0$  and then  $L \rightarrow \infty$  where  $N = N(L, \rho)$  is given by

$$N(L, \rho) = \frac{t}{\mathbb{P}(Z_{B_\rho(0)}^L \geq 1)/L} \sim tn^{\frac{1}{\gamma}}.$$

This contrasts with the Kac scaling which would behave like  $tn^{\frac{1}{\gamma}-1}$ .

## Proof of the theorem

One approximates the CP distribution by a compound Binomial distributed random variable

$$\tilde{W} = \sum_{j=0}^B Y_j,$$

where  $Y_j, j = 1, 2, \dots$ , are i.i.d. and  $B$  is binomial with  $(N', p)$ .

One uses  $N' = N/L$  and  $p = \mathbb{P}(Z_\rho^L \geq 1)$ .

Also  $\lambda_\ell(L, \rho) = \frac{1}{p} \mathbb{P}(Z_\rho^L = \ell)$ .

If  $N' = \frac{t}{\mathbb{P}(Z_\rho^L \geq 1)}$  then  $N'p \rightarrow t$  as  $\rho \rightarrow 0$  and  $B$  converges to

Poisson( $t$ ). As a result  $\tilde{W} \rightarrow CP$ .

For the central lemma let  $(X_n)_{n \in \mathbb{N}}$  be a stationary  $\{0, 1\}$ -valued process and put  $Z^L = \sum_{i=0}^{L-1} X_i$  for  $L \in \mathbb{N}$ . Let  $W_a^b = Z^b - Z^a = \sum_{i=a}^{b-1} X_i$  for  $0 \leq a < b$  and  $W = Z^N$ . (assume for simplicity's sake that  $N'$  and  $\Delta$  are integers.) Let  $L \ll N$  and denote by  $\tilde{\nu}$  be the CB distribution measure where the binomial part has values  $p = \mathbb{P}(Z^L \geq 1)$  and  $N' = N/L$  and the compound part has probabilities  $\lambda_\ell = \mathbb{P}(Z^L = \ell)/p$ .

## Lemma

For any  $\Delta \ll N'$

$$|\mathbb{P}(W = k) - \tilde{\nu}(\{k\})| \lesssim N'(\mathcal{R}_1 + \mathcal{R}_2) + \Delta \mathbb{P}(Z^L \geq 1),$$

where

$$\mathcal{R}_1 = \sup_{\substack{\Delta < M \leq N' \\ 0 < q < N' - \Delta}} \left| \sum_{u=1}^{q-1} \left( \mathbb{P}(Z^L = u \wedge W_{\Delta L}^{ML} = q - u) - \mathbb{P}(Z^L = u) \mathbb{P}(W_{\Delta L}^{ML} = q - u) \right) \right|$$

$$\mathcal{R}_2 = \sum_{j=2}^{\Delta} \mathbb{P}(Z^L \geq 1 \wedge W_{jL}^{(j+1)L} \geq 1).$$

The  $\mathcal{R}_1$  estimate uses decay of correlations.

Approximate  $\chi_{\mathcal{U}}$ ,  $\mathcal{U} = \{Z_0 = u\}$ , by Lipschitz functions from above and below.  $\mathcal{U} = \{Z_0 = u\}$  For  $r \ll \rho$  put  $U''(r) = B_r(\Gamma_\rho)$  (outer approximation of  $\Gamma_\rho$ ) and  $U'(r) = (B_r(\Gamma_\rho^c))^c$  (inner approximation).

The disjoint union

$$\mathcal{U} = \bigcap_{j=1}^u T^{-v_j} \Gamma_\rho \cap \bigcap_{i \in [0, L] \setminus \{v_j : j\}} T^{-i} \Gamma_\rho^c$$

$(0 \leq v_1 < v_2 < \dots < v_u \leq L - 1)$  counts all possible  $u$  entry time.

Outer approximation  $U''(r)$  then with  $U''(r)$  instead of  $\Gamma_\rho$  and inner approximation  $U'(r)$  with  $U'(r)$  instead of  $\Gamma_\rho$ .

Approximate  $\chi_{\mathcal{U}}$  from inside and outside by

$$\phi_r(x) = \begin{cases} 1 & \text{on } \mathcal{U} \\ 0 & \text{outside } U''(r) \end{cases} \quad \text{and} \quad \hat{\phi}_r(x) = \begin{cases} 1 & \text{on } U'(r) \\ 0 & \text{outside } \mathcal{U} \end{cases}.$$

With  $a = \sup_x |DT(x)|$  the Lipschitz norms of both  $\phi_r$  and  $\hat{\phi}_r$  are bounded by  $a^L/r$ . By construction  $\hat{\phi}_r \leq \chi u \leq \phi_r$  and also

$$\mathcal{U}''(r) \setminus \mathcal{U}'(r) \subset \bigcup_{i=0}^{L-1} T^{-i}(\Gamma_{\rho+r} \setminus \Gamma_{\rho-r}).$$

By the annulus assumption

$$\mu(\mathcal{U}''(r) \setminus \mathcal{U}'(r)) \lesssim L \frac{r^\xi}{\rho^\beta}.$$

For decay of correlations put  $r = \rho^w$  for  $w > 1$  a parameter.  
Then

$$\mathcal{R}_1 \lesssim a^L \frac{\mathfrak{e}(\Delta)}{\rho^w} + L\rho^{\xi w - \beta} \mathbb{P}(Z_\rho^L \geq 1).$$

Estimate of  $\mathcal{R}_2$ :

Put

$$\mathcal{C}(\Gamma_\rho) = \{\zeta : \zeta \text{ } n\text{-cylinder}, \zeta \cap \Gamma_\rho \neq \emptyset\}$$

is the outside  $n$ -cylinder approximation of the set  $\Gamma_\rho$

Put  $\mathcal{V} = \{Z_\rho^L \geq 1\}$  and estimate

$$\begin{aligned} \mu(T^{-j}\mathcal{V} \cap \Gamma_\rho) &\leq \sum_{\zeta \in \mathcal{C}_j(\Gamma_\rho)} \frac{\mu(T^{-j}\mathcal{V} \cap \zeta)}{\mu(\zeta)} \mu(\zeta) \\ &\lesssim \sum_{\zeta \in \mathcal{C}_j(\Gamma_\rho)} \mathfrak{D}(j) \frac{\mu(\mathcal{V} \cap T^j\zeta)}{\mu(T^j\zeta)} \mu(\zeta). \end{aligned}$$

The sets  $\zeta = \zeta_\varphi$  are  $\varphi$ -pre-images of  $R$ -balls, therefore the denominator is uniformly bounded from below because

$$\mu(T^j\zeta) = \mu(B_R(y_k)) > \text{const..}$$

Thus

$$\mu(T^{-j}\mathcal{V} \cap \Gamma_\rho) \lesssim \mathfrak{D}(j)\mu(\mathcal{V}) \sum_{\zeta \in \mathcal{C}_j(\Gamma_\rho)} \mu(\zeta) \lesssim \mathfrak{D}(j)\mu(\mathcal{V}) \mu\left(\bigcup_{\zeta \in \mathcal{C}_j(\Gamma_\rho)} \zeta\right).$$

Now

$$\bigcup_{\zeta \in \mathcal{C}_j(\Gamma_\rho)} \zeta \subset B_{\delta(j)}(\Gamma_\rho)$$

and by dimension assumption  $\mu(B_{\delta(j)}(U)) \approx (\delta(j) + \rho)^{d_0}$  we get

$$\mu(T^{-j}\mathcal{V} \cap \Gamma_\rho) \lesssim \mathfrak{D}(j)\mu(\mathcal{V})(j^{-\mathfrak{k}d_0} + \rho^{d_0})$$

( $\mathfrak{k}$  decay rate of  $\delta$ ).

Therefore, if  $j \geq 2$

$$\mu(\mathcal{V} \cap T^{-jL}\mathcal{V}) \leq \sum_{u=(j-1)L}^{jL-1} \mu(T^{-j}\mathcal{V} \cap \Gamma_\rho)$$

and some more careful estimate for  $j = 1$  then yields an estimate for  $\mathcal{R}_2$  (in the case when  $\mathfrak{D}(j) = \mathcal{O}(1)$ ):

$$\begin{aligned} \mathcal{R}_2 &\leq \sum_{j=1}^{\Delta} \mu(\mathcal{V} \cap T^{-jL}\mathcal{V}) \\ &\lesssim \mu(\mathcal{V})(L^{-\alpha t d_0} + L\Delta\rho^{d_0}) + \mu(\mathcal{V}') \end{aligned}$$

for some  $\alpha \in (0, 1)$  where  $\mathcal{V}' = \{Z_\rho^{L^\alpha} \geq 1\}$  (to accomodate  $j = 1$ ).

Since  $N' = t/\mu(\mathcal{V})$

$$N'\mathcal{R}_2 \lesssim L^{-v_1} + L\rho^{v_2} + \frac{\mu(\mathcal{V}')}{\mu(\mathcal{V})}$$

for some  $v_1, v_2 > 0$ . This allows us to execute the double limit of first  $\rho \rightarrow 0$  and then  $L \rightarrow \infty$  where

$$\lim_{L \rightarrow \infty} \lim_{\rho \rightarrow 0} \frac{\mu(\mathcal{V}')}{\mu(\mathcal{V})} = \lim_{L \rightarrow \infty} \lim_{\rho \rightarrow 0} \frac{\mathbb{P}(Z_\rho^{L^\alpha} \geq 1)}{\mathbb{P}(Z_\rho^L \geq 1)} = 0.$$