Height Function Transformations of Minimal Graph Surfaces

ISQGD–SS04 : Analysis on Manifolds

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Outline

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Minimal Graph Surfaces

Definition

Let Ω be an open connected set in $\mathbb{R}^2$ and let $f : \Omega \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a smooth function. Then f is called a **Minimal Graph Surface** if the graph of f forms a minimal surface in the Euclidean space $\mathbb{E}^3$.
Alternatively, a minimal graph surface can also be defined using the following partial differential equation, known as the minimal surface equation:

$$(1 + f_y^2) f_{xx} - 2 f_{xy} f_x f_y + (1 + f_x^2) f_{yy} = 0 \tag{1}$$

Examples

Some basic examples of minimal graph surfaces are the following:

1. **Plane:** $f(x,y) = ax + by + c$
2. **Helicoid:** $f(x,y) = \tan^{-1}\left(\frac{y}{x}\right), \quad x \neq 0$
3. **Catenoid:** $f(x,y) = \cosh^{-1}\left(\sqrt{x^2+y^2}\right), \quad x^2+y^2 \geq 1$
4. **Scherk's First Surface:** $f(x,y) = \ln\left(\frac{\cos y}{\cos x}\right), \quad |x| < \frac{\pi}{2},\ |y| < \frac{\pi}{2}$
5. **Scherk's Second Surface:** $f(x,y) = \sin^{-1}(\sinh(x)\sinh(y))$, where $|\sinh(x)\sinh(y)| \leq 1$

Minimal Graph Transformations

Definition

A smooth map $g : U \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is called a **minimal graph transformation** of the minimal graph surface $f : \Omega \longrightarrow \mathbb{R}$ if $f(\Omega) \subseteq U$ and the composition $g \circ f : \Omega \longrightarrow \mathbb{R}$ is again a minimal graph surface.

Definition

- A map $g : \mathbb{R} \longrightarrow \mathbb{R}$ is called a **trivial minimal graph transformation** (or **TMG transformation**) if for every minimal graph surface $f : \Omega \longrightarrow \mathbb{R}$, the composition $g \circ f$ is again a minimal graph surface.
- A minimal graph transformation of a minimal graph surface which is not trivial is called a **nontrivial minimal graph transformation** (or **NMG transformation**).

Problem Statement

- Explicitly find all the trivial minimal graph transformations.
- Explicitly find all the nontrivial minimal graph transformations and their corresponding minimal graph surfaces.

Outline

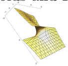
MINIMAL GRAPH TRANSFORMATIONS

→ Ruling out Trivial Cases → TMG Transformation and Constant Minimal Graph Surfaces

↓ Nontrivial Minimal Graph Transformations of Nonconstant Minimal Surfaces

Minimal Surface Equation + $\frac{\Delta f}{\|\nabla f\|^2} = h(f)$

→ First Characteristic Function h is zero → Helicoids and Planes



NMG Transformation: Parameter Change

↓ Assume $h \neq 0$: Conversion to an Equivalent system

$$\frac{u_{zz}u_{\bar{z}}^2 + u_{\bar{z}\bar{z}}u_z^2}{u_z u_{\bar{z}}} = j(u)$$

and

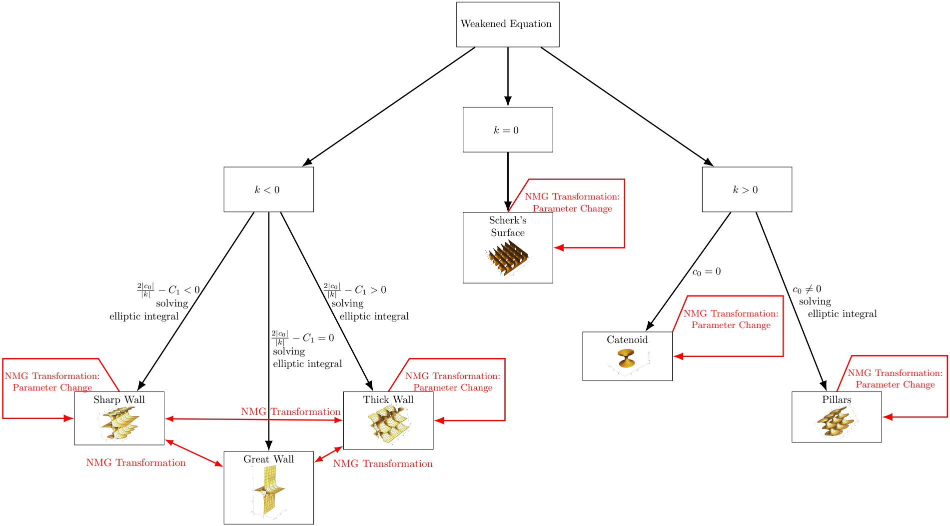
$$\Delta u = 0$$

↓ Weakening Technique

Weakened Eqn

$$[\ln(A'(z))]_{zz} = k[A'(z)]^2$$

where $k \in \mathbb{R}$


Weakened Equation
$k = 0$
$k < 0$
$k > 0$
NMG Transformation: Parameter Change
Scherk's Surface
$\frac{2|c_0|}{|k|} - C_1 < 0$ solving elliptic integral
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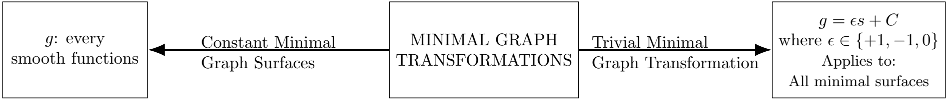
$\frac{2|c_0|}{|k|} - C_1 = 0$ solving elliptic integral
$c_0 = 0$
$c_0 \neq 0$ solving elliptic integral
Catenoid
NMG Transformation: Parameter Change
NMG Transformation: Parameter Change
Sharp Wall
NMG Transformation
NMG Transformation: Parameter Change
Thick Wall
NMG Transformation: Parameter Change
Pillars
Great Wall
NMG Transformation
NMG Transformation

Outline


g: every smooth functions
Constant Minimal Graph Surfaces
MINIMAL GRAPH TRANSFORMATIONS
Trivial Minimal Graph Transformation
$g = \epsilon s + C$
where $\epsilon \in \{+1, -1, 0\}$
Applies to:
All minimal surfaces

Trivial Cases

TMG Transformations

- We have seen that TMG transformations are precisely the maps which transform every minimal graph surface into another minimal graph surface.
- It can be shown that g is a TMG transformation if and only if $g'^3 - g' = 0$; in other words, $g(t) \equiv C$, $g(t) = t + C$, or $g(t) = -t + C$ for some constant $C \in \mathbb{R}$.

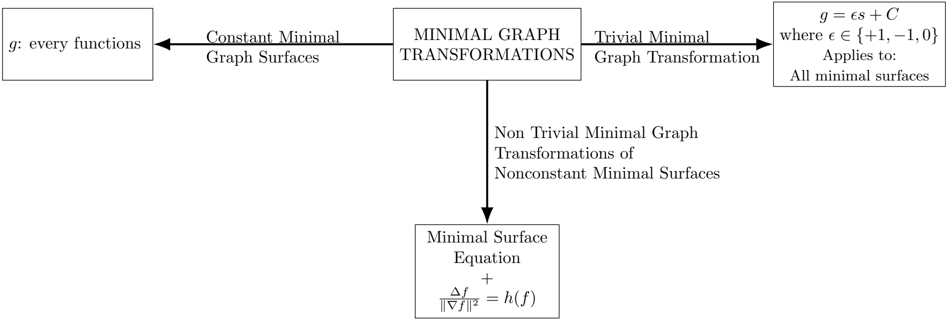
Constant Minimal Graph Surfaces

- One can also ask: for which minimal graph surfaces does every smooth function g yield a minimal graph transformation?
- It can be shown that the only minimal graph surfaces for which every smooth function is a minimal graph transformation must be constant minimal graph surfaces.
- The converse is also true: the set of all minimal graph transformations of any constant minimal graph surface is precisely the set of all smooth functions.

Choice of Domains

1. We have dealt with TMG transformations where $g'^3 - g' \equiv 0$ and constant minimal graph surfaces where $\|\nabla f\|^2 \equiv 0$. Now we will deal with NMG transformations of nonconstant minimal graph surfaces.
2. It is a well-known fact that all minimal graph surfaces are real analytic functions. Therefore, by the identity theorem for real analytic functions, it suffices to solve the problem on a smaller, well-behaved domain.
3. Specifically, a well-behaved subdomain $\Omega_0 \subseteq \Omega$ means that for the minimal graph surface defined on Ω_0, both $\|\nabla f\|^2$ and $(g'^3 - g')(f)$ are nonvanishing. We then solve the problem on this subdomain.
4. It can be shown that such a domain $\Omega_0 \subseteq \Omega$ exists. Hence, our strategy is to first find solutions on a suitable subdomain Ω_0 and then extend them globally via analytic continuation.

Outline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Minimal Surface Equation
+
$\frac{\Delta f}{\|\nabla f\|^2} = h(f)$

Nontrivial Minimal Graph Transformation Problem

- We have seen that $g'^3 - g' \not\equiv 0$ and $\|\nabla f\|^2 \not\equiv 0$ on some well-behaved domain Ω_0. This, along with the fact that f is a minimal graph surface, gives us the following pair of PDEs:

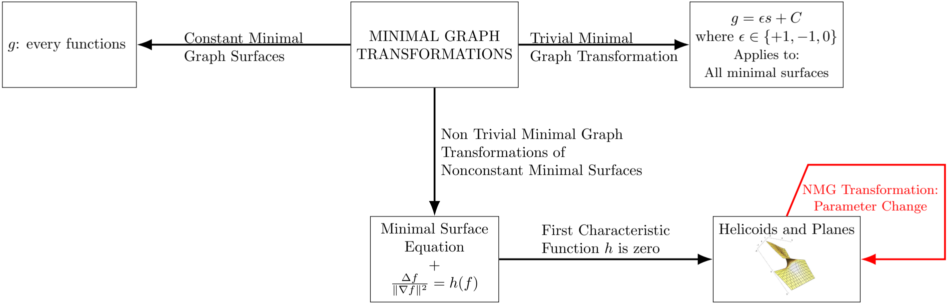
$$\frac{\Delta f}{\|\nabla f\|^2}\Big|_{(x,y)} = \frac{g''}{g'^3 - g'}\Big|_{f(x,y)} \qquad (1 + f_y^2) f_{xx} - 2 f_{xy} f_x f_y + (1 + f_x^2) f_{yy} = 0.$$

- We denote the function $\frac{g''}{g'^3 - g'}$ by h. Given h, we can find the minimal graph transformation g via the following formula:

$$g = \pm \int \frac{1}{\sqrt{1 \pm C^2 e^{2\int h\, dt}}}\, dt + D \tag{2}$$

- Thus we obtain the following pair of PDEs, which we call the NMG transformation problem. The function h will be referred to as the first characteristic function of the NMG transformation problem:

$$\frac{\Delta f}{\|\nabla f\|^2} = h(f) \qquad (1 + f_y^2) f_{xx} - 2 f_{xy} f_x f_y + (1 + f_x^2) f_{yy} = 0 \tag{3}$$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First Characteristic Function h is zero
Helicoids and Planes
NMG Transformation: Parameter Change

Zero First Characteristic Function

- First, convert the NMG transformation problem to complex coordinates $(z, \bar{z})$:

$$f_{zz} f_{\bar{z}}^2 + f_{\bar{z}\bar{z}} f_z^2 - f_{z\bar{z}}(1 + 2 f_z f_{\bar{z}}) = 0 \quad \text{and} \quad \frac{f_{z\bar{z}}}{f_z f_{\bar{z}}} = h(f) \tag{4}$$

- Assume the First Characteristic Function $h = 0$. Then system 4 reduces to:

$$f_{zz} f_{\bar{z}}^2 + f_{\bar{z}\bar{z}} f_z^2 = 0 \quad \text{and} \quad f_{z\bar{z}} = 0$$

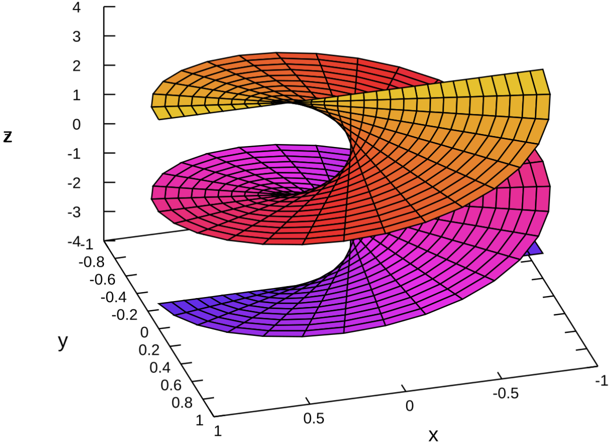
- Solve this system. Then compute g using formula 2. We obtain:

$$f(x, y) = \frac{-2}{\alpha} \tan^{-1}\left(\frac{y+\gamma}{x+\beta}\right) + \delta \quad \text{or} \quad f(x, y) = \alpha x + \beta y + \gamma$$

  and $g(t) = at + b$. Here $a, b, \alpha, \beta, \gamma, \delta \in \mathbb{R}$ are arbitrary constants.

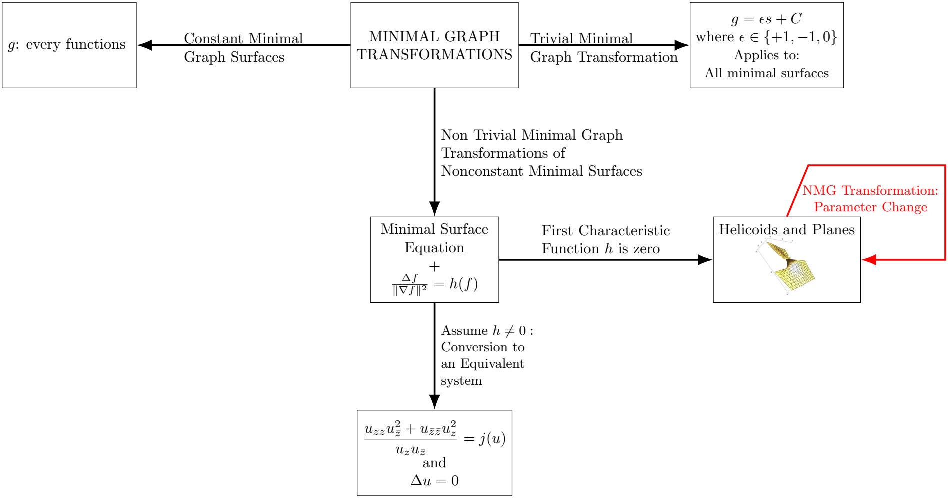
- The resulting minimal graph surfaces are **helicoids and planes**. Under g, a helicoid transforms to another helicoid. Similarly, a plane transforms to another plane. The parameters are changed, but the type is preserved.

Helicoids



Additional Conditions on Choice of Domain

- We explicitly characterized all minimal graph surfaces for which the first characteristic function h is identically zero.
- Now we operate under the assumption that h is not the zero function, hence we now restrict our domain Ω to a sufficiently small region such that $h(f)$ is non-vanishing on Ω.
- Note that the domain restrictions introduced here do not affect the global nature of our analysis. Since solutions to the minimal surface equation are real analytic, any solution defined on a small open set Ω possesses a unique analytic continuation to a maximal domain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Assume $h \neq 0$: Conversion to an Equivalent system
$\frac{u_{zz}u_{\bar{z}}^2 + u_{\bar{z}\bar{z}}u_z^2}{u_z u_{\bar{z}}} = j(u)$
and
$\Delta u = 0$

Strategy for solving the NMG Transformation Problem

Conversion to a simpler system

- The pair (f, h) solves the nontrivial minimal graph transformation problem 4. Here, $h : I \to \mathbb{R}$ is the first characteristic function, which is non-vanishing and sign-definite on $I = \mathrm{Range}(f)$.
- We apply the following change of variables:

$$H(t) = \int e^{-\int h}, \quad K = H^{-1}, \quad u = H \circ f, \quad j(s) = \frac{h(K(s))}{K'(s)}.$$

- Then the functions u and j satisfy the following modified PDEs:

$$u_{z\bar{z}} = 0 \quad \text{and} \quad \frac{u_z^2 u_{\bar{z}\bar{z}} + u_{\bar{z}}^2 u_{zz}}{u_z u_{\bar{z}}} = j(u). \tag{5}$$

This system will be referred to as the modified nontrivial minimal graph transformation problem, and the function j will be called the second characteristic function of the NMG transformation problem.

Strategy for solving the NMG Transformation Problem

Equivalence of both systems: Setup for the Theorem

System 1: The pair (f, h) solves the Non-Trivial Minimal Graph Transformation Problem 4, where:

- $f : \Omega \to \mathbb{R}$ is a minimal graph surface with $f_z, f_{\bar{z}} \neq 0$
- $h : I \to \mathbb{R}$ is the first characteristic function, non-vanishing and sign-definite on $I = \text{Range}(f)$

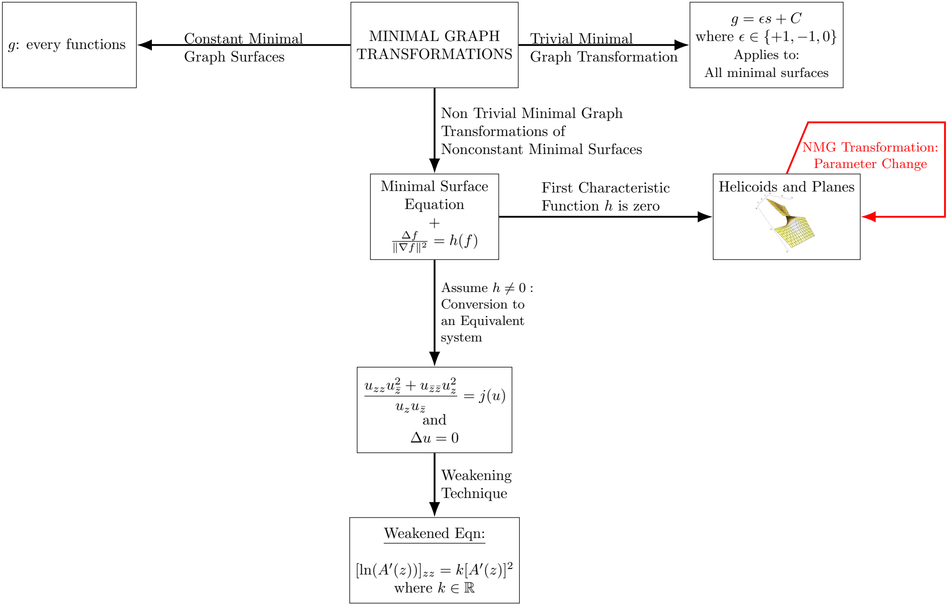
System 2: The pair (u, j) solves the Modified Non-Trivial Minimal Graph Transformation Problem 5, where:

- $u : \Omega \to \mathbb{R}$ is harmonic with $u_z, u_{\bar{z}} \neq 0$
- $j : J \to \mathbb{R}$ is the second characteristic function, non-vanishing and sign-definite on $J = u(\Omega)$

Strategy for solving the NMG Transformation Problem

Equivalence Theorem:

- It can be shown that the two systems—the NMG transformation problem and the modified NMG transformation problem—are equivalent in the following sense:
- Given any solution (f, h) of the NMG transformation problem 4, there exists a solution (u, j) of the modified NMG transformation problem 5 from which one can construct (f, h) via an explicit transformation.
- Conversely, given any solution (u, j) of the modified NMG transformation problem 5, there exists a solution (f, h) of the NMG transformation problem 4 from which one can construct (u, j) via an explicit inverse transformation, at least in a sufficiently small neighborhood around any point in its original domain.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and
$\Delta u = 0$
Weakening
Technique
Weakened Eqn:
$[\ln(A'(z))]_{zz} = k[A'(z)]^2$
where $k \in \mathbb{R}$

Strategy for solving the NMG Transformation Problem

Weakening

- Let us begin with the modified NMG transformation problem 5. The function u is a harmonic real-valued function, so it can be written as $u(z,\bar{z}) = A(z) + \overline{A(z)}$. Denote $B(\bar{z}) = \overline{A(z)}$.
- Substituting this into the system yields:

$$\frac{A''(z)[B'(\bar{z})]^2 + B''(\bar{z})[A'(z)]^2}{A'(z)B'(\bar{z})} = j(u).$$

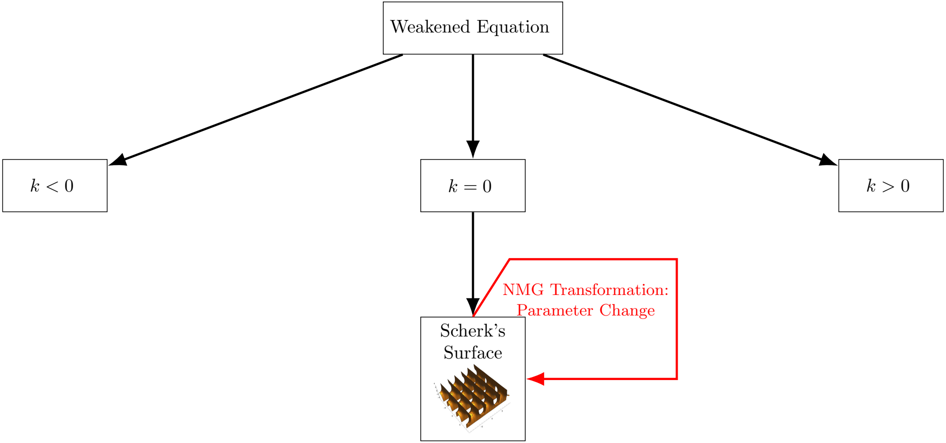
- Differentiating the RHS with respect to z and multiplying by $B'(\bar{z})$ gives $j'(u)A'(z)B'(\bar{z})$. Similarly, differentiating with respect to $\bar{z}$ and multiplying by $A'(\bar{z})$ also gives $j'(u)A'(z)B'(\bar{z})$. Performing the same operations on the LHS, equating the results, and simplifying yields:

$$[\ln(A'(z))]_{zz} = k[A'(z)]^2 \qquad \text{for a real constant } k. \tag{6}$$

Strategy for solving the NMG Transformation Problem

- This differentiation process yields a weaker system than the original 5, so it may admit additional solutions.
- Our strategy is to exhaustively solve Equation 6 for the three cases $k > 0$, $k = 0$, and $k < 0$, and then determine which of these solutions also satisfy the modified NMG transformation problem 5.
- This involves various identities involving elliptic functions and elliptic integrals.
- Due to time constraints, we will not go into the details of these calculations.
- Instead, we present the remaining exact solutions to the nontrivial minimal graph transformation problem.

Outline


Weakened Equation
$k < 0$
$k = 0$
$k > 0$
Scherk's Surface
NMG Transformation: Parameter Change

Case $k = 0$: Scherk's Surface

- We explicitly solve equation 6 for the case $k = 0$ and find that its solutions yield Scherk's surfaces, possibly with a rotated domain:

$$f = \frac{1}{|a|}\sin^{-1}\left[\frac{2\Re\left(\frac{1}{a}e^{az+b}\right)}{C_3^2}\right] - C_2$$

where $z = x + iy$, $a, b \in \mathbb{C}$ are arbitrary complex constants, and $C_2, C_3 \in \mathbb{R}$ are arbitrary real constants.

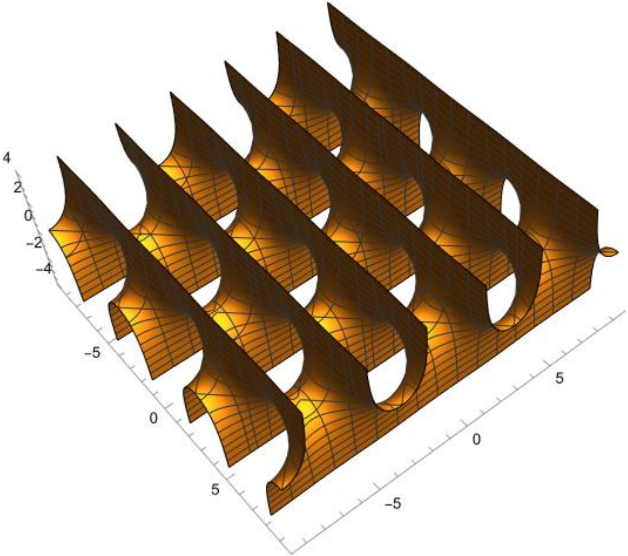
- The corresponding NMG transformation is given by:

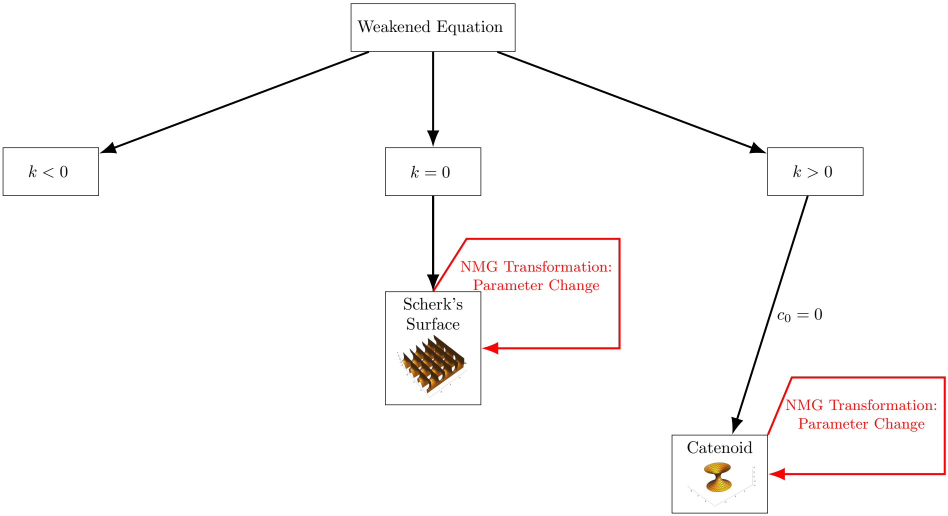
$$g(s) = \pm\frac{1}{|a|}\sin^{-1}\left(\frac{\sin[|a|(s + C_2)]}{C_5^2}\right) + D,$$

with $C_5, D \in \mathbb{R}$ arbitrary real constants.

- This transformation maps a Scherk's surface with one set of parameters to another Scherk's surface with different parameters.

Scherk's Surface




Weakened Equation
$k < 0$
$k = 0$
$k > 0$
Scherk's Surface
NMG Transformation: Parameter Change
$c_0 = 0$
Catenoid
NMG Transformation: Parameter Change

Case $k > 0$: Catenoid

- We explicitly solve equation 6 for the case $k > 0$ and find that there are two families of solutions. The first is the catenoid surface family given by:

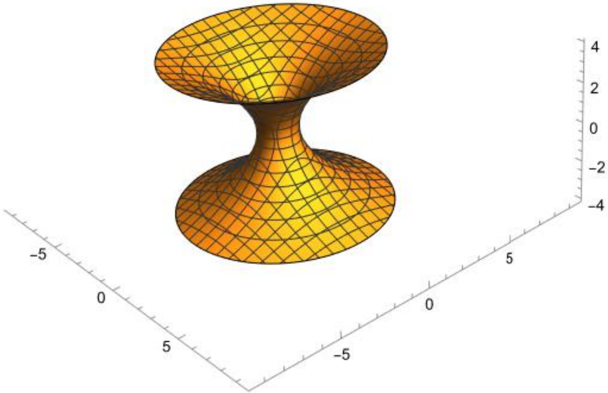
$$f = \frac{\epsilon_0}{\lambda} \cosh^{-1}\left[\lambda |c_1 + z|\right] - C_2$$

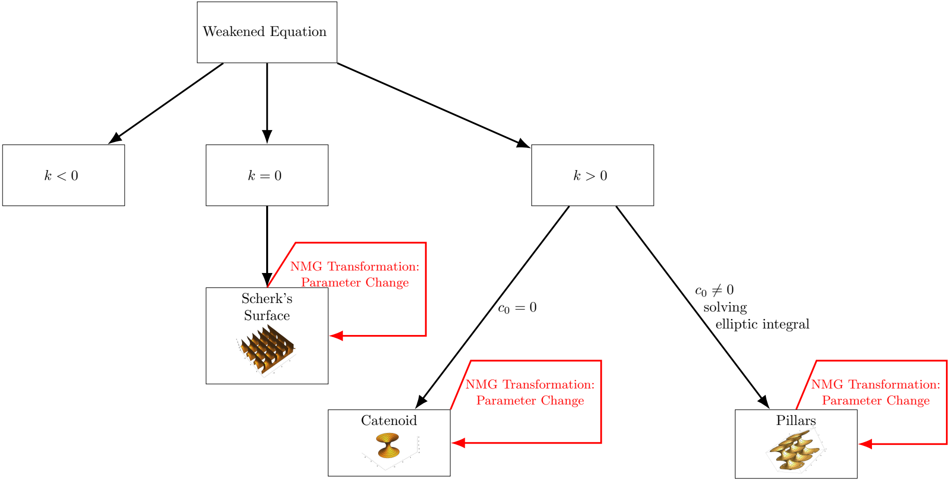
  where $z = x + iy$, $c_1 \in \mathbb{C}$ is an arbitrary complex constant, and $\lambda, C_2 \in \mathbb{R}$ are arbitrary real constants.
- The corresponding NMG transformation is given by:

$$g(s) = \frac{\pm 1}{\lambda\mu} \cosh^{-1}\left(\mu \cdot \cosh[\lambda(s + C_2)]\right) + D$$

  with $\mu, D \in \mathbb{R}$ arbitrary real constants.
- This transformation maps a catenoid with one set of parameters to another catenoid surface with different parameters.

Catenoid

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$k > 0$
NMG Transformation: Parameter Change
Scherk's Surface
$c_0 = 0$
$c_0 \neq 0$ solving elliptic integral
NMG Transformation: Parameter Change
NMG Transformation: Parameter Change
Catenoid
Pillars

Case $k > 0$: Pillars

- The second family of surfaces is given by:

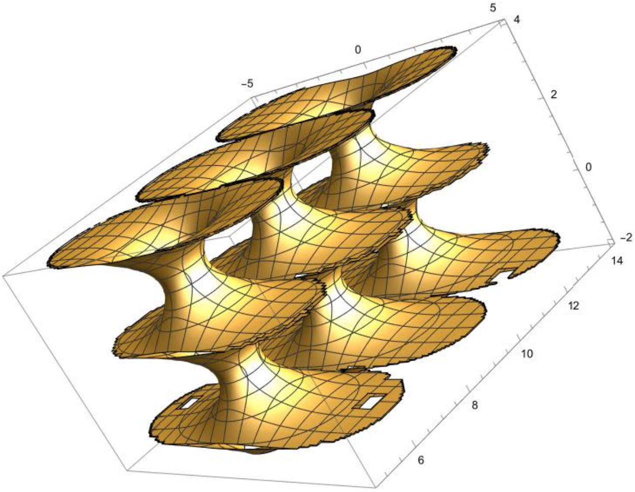
$$f = \frac{\pm\gamma'}{\sqrt{|c_0|}}\,\mathrm{cn}^{-1}\left[\frac{\gamma'}{\gamma}\sqrt{\cosh\left[2\ln\left|\tanh\left(\frac{\sqrt{c_0}(z+c_1)}{2}\right)\right|\right]-1}\,,\ \gamma\right]-C_2$$

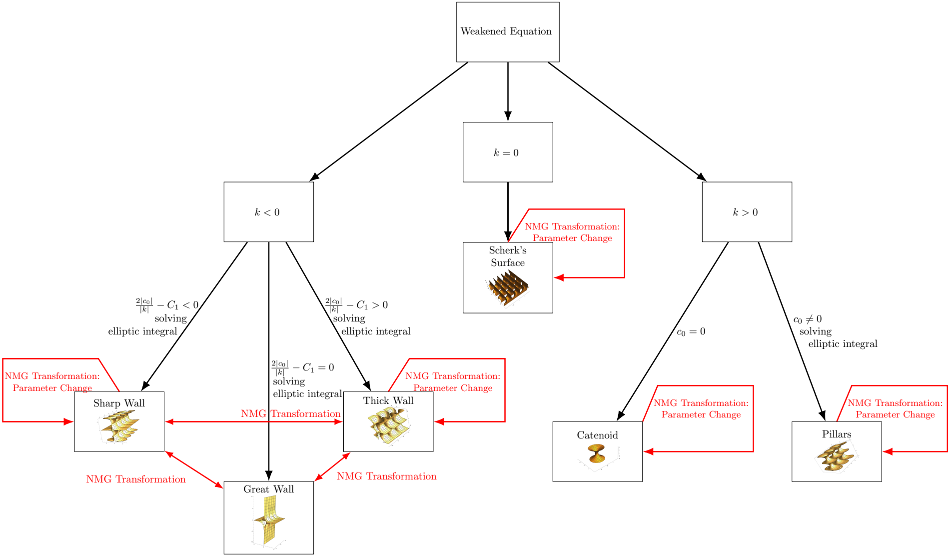
  where $z = x + iy$, $c_0, c_1 \in \mathbb{C}$ are arbitrary complex constants, and $C_2 \in \mathbb{R}$ are arbitrary real constants, and $\gamma \in [0,1]$ and $\gamma' = \sqrt{1-\gamma^2}$. We name this minimal surface **Pillars**.
- The corresponding NMG transformation is given by:

$$g = \frac{\pm\Lambda'}{\sqrt{|c_0|}}\,\mathrm{sd}^{-1}\left[\frac{\gamma}{\gamma'\Lambda}\,\mathrm{cn}\left(\frac{\sqrt{|c_0|}}{\gamma'}(s+C_2),\ \gamma\right),\ \Lambda\right]+D$$

  with $\Lambda, \Lambda' = \sqrt{1-\Lambda^2}, D \in \mathbb{R}$ arbitrary real constants.
- This transformation maps a Pillar surface with one set of parameters to another Pillar surface with different parameters.

Pillars

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NMG Transformation
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NMG Transformation: Parameter Change
Pillars
NMG Transformation: Parameter Change

Case $k < 0$

The case $k < 0$ yields three families of solutions. The first is defined by:

$$f = \frac{\epsilon_2}{\sqrt{|c_0|}} \operatorname{sech}^{-1}\left[\frac{1}{\sqrt{2}}\sqrt{1+\cosh\left(\ln\left[\tanh(\tfrac{\sqrt{c}_0(z+c_1)}{2})\coth(\tfrac{\sqrt{\bar{c}_0}(\bar{z}+\bar{c}_1)}{2})\right]\right)}\,\right] - C_2$$

The second family is given by:

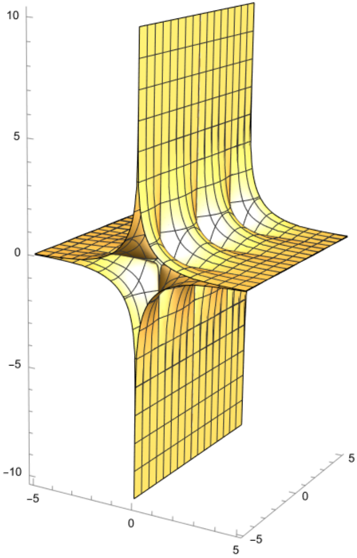
$$f = \frac{\epsilon_2}{\sqrt{|c_0|}} \operatorname{dn}^{-1}\left[\frac{1}{\sqrt{2}}\sqrt{1+\cosh\left(\ln\left[\tanh(\tfrac{\sqrt{c}_0(z+c_1)}{2})\coth(\tfrac{\sqrt{\bar{c}_0}(\bar{z}+\bar{c}_1)}{2})\right]\right)}, \gamma\right] - C_2$$

And finally, the third family is:

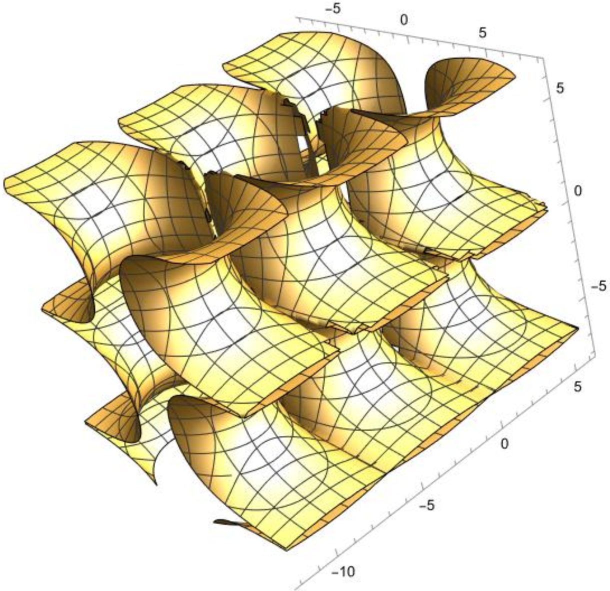
$$f = \epsilon_2 \frac{\gamma}{\sqrt{|c_0|}} \operatorname{cn}^{-1}\left[\frac{1}{\sqrt{2}}\sqrt{1+\cosh\left(\ln\left[\tanh(\tfrac{\sqrt{c}_0(z+c_1)}{2})\coth(\tfrac{\sqrt{\bar{c}_0}(\bar{z}+\bar{c}_1)}{2})\right]\right)}, \gamma\right] - C_2$$

where $z = x + iy$, $c_0, c_1 \in \mathbb{C}$ are arbitrary complex constants, and $C_2 \in \mathbb{R}$ is an arbitrary real constant and $\gamma \in [0, 1]$. We name these minimal surfaces **Great Wall**, **Thick Wall**, and **Sharp Wall**, respectively.

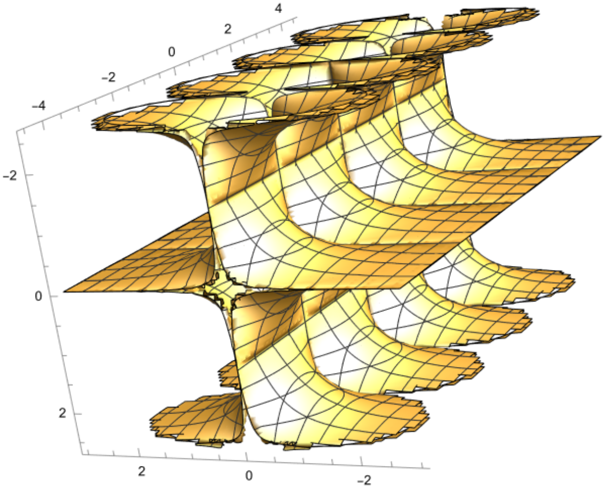
Great Wall



Thick Wall



Sharp Wall



NMG Transformation: Great Wall

There are two types of NMG transformations for the Great Wall, given by:

- $g_1 = \frac{\pm 1}{\sqrt{1+C_4^2}\sqrt{|c_0|}} \operatorname{cn}^{-1}\left[\operatorname{sech}\left[\sqrt{|c_0|}(s+C_2)\right], \frac{1}{\sqrt{1+C_4^2}}\right] + C_3$
- $g_2 = \frac{\pm 1}{\sqrt{|c_0|}} \operatorname{dn}^{-1}\left[\operatorname{sech}\left[\sqrt{|c_0|}(s+C_2)\right], \sqrt{1-C_4^2}\right] + C_3$

Here c_0 and C_2 are exactly the same as in the definition of the Great Wall, and $C_3, C_4 \in \mathbb{R}$ are arbitrary real constants.

- These are the NMG transformations that map the Great Wall to the Thick Wall and Sharp Wall, respectively.
- Note that there is no NMG transformation for this surface that performs a parameter change to map the Great Wall family back to itself, unlike in all previous cases.

NMG Transformation: Thick Wall

There are three types of NMG transformations for the Thick Wall, given by:

- $g_1 = \frac{\pm 1}{\sqrt{|c_0|}} \operatorname{sech}^{-1} \left[\operatorname{dn} \left[\sqrt{|c_0|}(s + C_2), \gamma \right] \right] + C_3$
- $g_2 = \frac{\pm 1}{\gamma C_4 \sqrt{|c_0|}} \operatorname{cn}^{-1} \left[\operatorname{dn} \left[\sqrt{|c_0|}(s + C_2), \gamma \right], \frac{1}{\gamma C_4} \right] + C_3$
- $g_3 = \frac{\pm 1}{\sqrt{|c_0|}} \operatorname{dn}^{-1} \left[\operatorname{dn} \left[\sqrt{|c_0|}(s + C_2), \gamma \right], C_4 \gamma \right] + C_3$

Here c_0 and C_2 are exactly the same as in the definition of the Thick Wall, and $C_3, C_4 \in \mathbb{R}$ are arbitrary real constants.

- These are the NMG transformations that map the Thick Wall to the Great Wall, Thick Wall, and Sharp Wall, respectively.
- Note that, unlike the Great Wall, there exists an NMG transformation for this surface that performs a parameter change to map the Thick Wall family back to itself.

NMG Transformation: Sharp Wall

There are three types of NMG transformations for the Sharp Wall, given by:

- $g_1 = \frac{\pm 1}{\sqrt{|c_0|}} \operatorname{sech}^{-1}\left[\operatorname{cn}\left(\frac{\sqrt{|c_0|}}{\gamma}(s+C_2), \gamma\right)\right] + C_3$
- $g_2 = \frac{\pm 1}{C_4\sqrt{|c_0|}} \operatorname{cn}^{-1}\left[\operatorname{cn}\left(\frac{\sqrt{|c_0|}}{\gamma}(s+C_2), \gamma\right), \frac{1}{C_4}\right] + C_3$
- $g_3 = \frac{\pm 1}{\sqrt{|c_0|}} \operatorname{dn}^{-1}\left[\operatorname{cn}\left(\frac{\sqrt{|c_0|}}{\gamma}(s+C_2), \gamma\right), C_4\right] + C_3$

Here c_0 and C_2 are the same as in the definition of the Sharp Wall, while $C_3, C_4 \in \mathbb{R}$ are arbitrary real constants.

- These NMG transformations map the Sharp Wall to the Great Wall, Thick Wall, and Sharp Wall, respectively.
- In contrast to the Great Wall, the Sharp Wall admits an NMG transformation that performs a parameter change, mapping the Sharp Wall family back to itself.

Outline

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