Prequantization of 0-Symplectic Stacks

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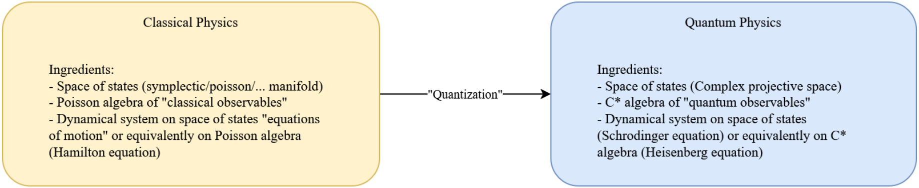
Contents

Introduction

Quantization

Quantization is a method in mathematical physics for constructing quantum mechanical models. There are several versions of quantization:

- Deformation quantization
- Geometric quantization
- Stochastic quantization
- Path integral quantization
- ...



Geometric Quantization

In this talk we focus on geometric quantization. In the usual setting, the first few steps of this method are,

1. Start with a classical space of states (e.g. a symplectic manifold (M, ω))
2. Construct the algebra of Hamiltonian vector fields $\mathrm{Ham}(M)$
3. ("Prequantization") Lift these vector fields to a principal $U(1)$-bundle $\pi : P \to M$ with curvature $\pi^*\omega$ to form a new algebra $\mathrm{Quant}(M)$

Kostant and Souriau's map from Hamiltonian vector fields (X, h) to quantum operators $\hat{X}_f$ is given by,

$$(X, h) \mapsto X^H + (2\pi)^{-1}(\pi^* h)\partial_\theta$$

This is called the quantization map.

Main Goal: Investigate prequantization for more general spaces (e.g. stacks).

Background

Lie Groupoids

The main object which we will try to quantize is a 0-Symplectic Lie groupoid[1]. Lie groupoids are a many-object generalization of Lie groups. They consist of

- Manifolds $\mathcal{G}$ (space of morphisms) and M (space of objects)
- Smooth submersions $s, t : \mathcal{G} \to M$ (source and target of morphisms)
- All morphisms smoothly invertible, and there exist identity morphisms $1_x : x \to x$ for all $x \in M$.
- Morphisms can be composed if their source and targets match.
- A 0-symplectic structure, which is a special type of 2-form on M (more on this to come)

0-Symplectic Lie groupoids defined by Pantev et al. [2013], studied by Hoffman and Sjamaar [2020] in depth. They present a type of symplectic stack.

[1] See e.g. Mackenzie [2005] for a reference on Lie groupoids and algebroids

Lie Algebroid of a Lie Groupoid

Just as a Lie group has a Lie algebra, a Lie groupoid has a Lie algebroid.

- It is a vector bundle $A \to M$
- There is a map $\rho : A \to TM$ called the anchor
- There is a Lie bracket on sections of A

Sections of A generate left and right-invariant vector fields on $\mathcal{G}$.

Foliation Groupoid

A particularly interesting class of Lie groupoids is the foliation groupoids - these are related to the study of leaf spaces of foliations and Poisson geometry.

Definition 1 (Foliation Groupoid)

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid with Lie algebroid A, having anchor map $\rho : A \to TM$.

We say $\mathcal{G}$ is a foliation groupoid if $\mathrm{im}(\rho) \subseteq TM$ is Frobenius integrable (i.e. is equal to the tangent distribution of a regular foliation $\mathcal{F}$).

In the case of foliation groupoids, the question of prequantization is particularly well-posed. Let us explain why.

Non-Degeneracy

Motivated by derived geometry, one defines an analogue of the tangent complex as,

$$\text{inclusion} : T\mathcal{F} \to TM$$

Definition 2 (Non-Degenerate 2-form (see Hoffman and Sjamaar [2020]))

Let $\mathcal{G} \rightrightarrows M$ be a foliation groupoid and let $\omega \in \Omega^2(M)$. We say that ω is non-degenerate if $\ker \omega = T\mathcal{F}$.

Main idea: the induced form on the leaf space (however pathological) is nondegenerate in the usual sense.

Definition 3 (0-Symplectic Form)

A 0-symplectic form is a non-degenerate 2-form $\omega \in \Omega^2(M)$ such that

$$\mathrm{d}\omega = 0 \quad \text{and} \quad s^*\omega = t^*\omega$$

Multiplicative Vector Fields

We also need a notion of Hamiltonian vector fields on 0-symplectic groupoids. We consider multiplicative vector fields (Mackenzie and Xu [1998]).

Definition 4 (Multiplicative Vector Field)

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. A multiplicative vector field is a pair $(X, Y) \in \mathfrak{X}(M) \times \mathfrak{X}(\mathcal{G})$ which is a functor section of $T\mathcal{G} \rightrightarrows TM$ (i.e. compatible with all structure maps). For example, one requires

$$\text{e.g.} \quad \mathrm{d}s(Y|_f) = X|_{s(f)}, \qquad \mathrm{d}t(Y|_f) = X|_{t(f)}, \qquad \forall f \in \mathcal{G}$$

Let us denote the set of all such pairs by $\mathfrak{X}_{\mathrm{mult}}(\mathcal{G})$.

This set forms a real Lie algebra with bracket

$$[(X, Y), (X', Y')] = ([X, X'], [Y, Y'])$$

Lie 2 Algebra of Vector Fields

Just as the space of vector fields on a manifold forms a Lie algebra, the set of multiplicative vector fields can be used to form a Lie 2-algebra.

Definition 5 (Lie 2-Algebra of Vector Fields)

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid with Lie algebroid A. One can define a crossed module of Lie algebras (a strict Lie 2-algebra) given by the complex,

$$\Gamma(A) \xrightarrow{\mathrm{d}^A} \mathfrak{X}_{\mathrm{mult}}(\mathcal{G})$$

where the map d^A is a Lie algebra morphism, given by $a \mapsto (\rho(a), a_L + a_R)$

This complex models the space of vector fields on the underlying stack (Berwick-Evans and Lerman [2020], Ortiz and Waldron [2019]).

Results

Assumptions

In what follows, let us assume the following

- The Lie groupoid $\mathcal{G} \rightrightarrows M$ is a foliation groupoid
- There exists a 0-symplectic form $\omega \in \Omega^2(M)$

Hamiltonian Vector Fields

Definition 6 (Hamiltonian Vector Fields (Hoffman and Sjamaar [2020]))

Let ω be a 0-symplectic form on $\mathcal{G} \rightrightarrows M$. A pair $(X, Y) \in \mathfrak{X}_{\mathrm{mult}}(\mathcal{G})$ is said to be Hamiltonian if,

$$\iota_X \omega = \mathrm{d}f$$

and

$$\iota_Y s^* \omega = \mathrm{d}g$$

for some functions $f \in C^\infty(M)^{\mathcal{G}}$, $g \in C^\infty(\mathcal{G})$. Denote the set of all such pairs by $\mathrm{Ham}_{\mathrm{mult}}(\mathcal{G})$.

Note: The fact that $\iota_Y s^* = s^* \iota_X$ implies $s^* f = t^* f = g$.

Hamiltonian Vector Fields - II

Proposition 7 (A. & D. Krepski)

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid with 0-symplectic form ω. Then the complex

$$\Gamma(A) \to \mathrm{Ham}_{\mathrm{mult}}(\mathcal{G})$$

is a strict Lie 2-algebra. We denote it $\mathcal{H}_\bullet$.

Proof.

Since ω is nondegenerate, $\iota_{\rho(a)}\omega = 0$, so all elements of $\mathrm{im}(\mathrm{d}^A)$ are Hamiltonian with $f = \mathrm{const}$. Thus this is a sub-2-algebra of the Lie 2-algebra of multiplicative vectors. □

Prequantum Bundle

Recall that in prequantization we wish to lift the Hamiltonian vector fields from M to a prequantum bundle $P \to M$.

Definition 8

Let $\mathcal{G} \to M$ be a 0-symplectic groupoid. An equivariant prequantum bundle $\pi : P \to M$ is a $U(1)$-principal bundle, equipped with an action of $\mathcal{G}$ for which π is equivariant, and with connection γ satisfying $\mathrm{d}\gamma = \pi^*\omega$ and $s^*\gamma = t^*\gamma$.

We summarize this with an action groupoid:

$$\begin{array}{ccc} \mathcal{G} \ltimes P & \xrightarrow{\mathrm{pr}_1} & \mathcal{G} \\ {\scriptstyle \tilde{s}}\downarrow\downarrow{\scriptstyle \tilde{t}} & & {\scriptstyle s}\downarrow\downarrow{\scriptstyle t} \\ P & \xrightarrow[\pi]{} & M \end{array}$$

Horizontal Lift

Proposition 9 (A. & D. Krepski)

Let (X, Y) be a Hamiltonian vector field on $\mathcal{G} \rightrightarrows M$ and let $P \to M$ be an equivariant prequantum bundle with connection γ. Then there exists a unique horizontal lift of (X, Y), i.e. a pair

$$(\tilde{X}, \tilde{Y}) \in \mathfrak{X}_{\mathrm{mult}}(\mathcal{G} \ltimes P)$$

satisfying

$$\mathrm{d}\pi(\tilde{X}) = X, \qquad \mathrm{dpr}_1(\tilde{Y}) = Y$$

and

$$\gamma(\tilde{X}) = 0 \qquad \text{(horizontal)}$$

Poisson Algebra

Proposition 10 (A. & D. Krepski)

Consider the space of tuples,

$$\mathrm{Pois}(\mathcal{G}) = \{(X, Y, f) : \mathrm{d}f = \iota_X \omega, \mathrm{d}s^* f = \iota_Y s^* \omega\}$$

Then one can construct a complex,

$$\mathrm{Pois}_\bullet(\mathcal{G}) : \quad \Gamma(A) \to \mathrm{Pois}(\mathcal{G})$$

This forms a **2-term homotopy Poisson algebra** with structure,

$$\begin{aligned}
\{(X_1, Y_1, f_1), (X_2, Y_2, f_2)\} &= ([X_1, X_2], [Y_1, Y_2], -\iota_{X_1}\iota_{X_2}\omega), \\
\{(X, Y, f), a\} &= ([(X, Y), a], 0), \\
\{a_1, a_2\} &= [a_1, a_2] \\
(X_1, Y_1, f_1) \cdot (X_2, Y_2, f_2) &= (f_1 X_2 + f_2 X_1, s^* f_1 Y_2 + s^* f_2 Y_1, f_1 f_2), \\
(X_1, Y_1, f_1) \cdot a_1 &= (s^* f_1) a_1, \\
a_1 \cdot a_2 &= 0.
\end{aligned}$$

Main Theorem I

Let $\mathcal{H}_\bullet = \Gamma(A) \to \mathrm{Ham}_{\mathrm{mult}}(\mathcal{G})$ be the Lie 2-algebra of Hamiltonian vector fields.
Let $\mathcal{O}_\bullet = \mathbb{R} \to C^\infty(M)^{\mathcal{G}}$ be the Lie 2-algebra of invariant functions[2]

Theorem 11 (A. & D. Krepski)

The homotopy Poisson algebra $\mathrm{Pois}_\bullet(\mathcal{G})$ defines a generalized morphism of Lie 2-algebras,

$$\mathrm{Pois}_\bullet(\mathcal{G}) : \mathcal{O}_\bullet \dashrightarrow \mathcal{H}_\bullet$$

which can be implemented using e.g. Noohi's 2-butterflies (see Noohi [2013]).

[2] the differential is simply the inclusion of constant functions

Kostant-Souriau Central Extension

Corollary 12 (A. & D. Krepski)

Suppose M is connected. The sequence of chain complexes,

$$\operatorname{hfib}(\operatorname{Pois}_\bullet(\mathcal{G})) \dashrightarrow (\mathcal{O}_\bullet)_{>1} \dashrightarrow \mathcal{H}_\bullet \dashrightarrow \operatorname{hfib}(\operatorname{Pois}_\bullet(\mathcal{G}))[1] \tag{1}$$

is an exact triangle (i.e. an exact sequence up to homotopy).

Moreover, if $\mathcal{G} \rightrightarrows M$ is a manifold (i.e. $\mathcal{G} \cong M \times M$), the induced long exact sequence in homology reduces to the classical central extension result of Kostant-Souriau,

$$\mathbb{R} \to C^\infty(M) \to \operatorname{Ham}(M) \tag{2}$$

Quantum Lie 2-Algebra

Definition 13 (Infinitesimal Quantomorphisms)

An infinitesimal quantomorphism of the action groupoid $\mathcal{G} \ltimes P$ is a multiplicative vector field $(\tilde{X}, \tilde{Y}) \in \mathfrak{X}(P_\bullet)$ satisfying $\mathcal{L}_{\tilde{X}}\gamma = 0$.

We denote the subspace of all such vector fields by $\mathrm{Quant}(P, \gamma)$.

Proposition 14 (A. & D. Krepski)

The complex $\mathcal{Q}_\bullet : \Gamma(\pi^! A) \to \mathrm{Quant}(P, \gamma)$ given by $\mathrm{d}^P(a^!) = (\rho_P(a^!), a^!_L + a^!_R)$ is a strict Lie 2-algebra.

We call this the Lie 2-algebra of infinitesimal quantomorphisms, and define a quantization map,

$$Q : \mathrm{Ham}_{\mathrm{mult}}(\mathcal{G}) \to \mathrm{Quant}(P, \gamma)$$

analogously to the classical case.

Main Theorem II

Theorem 15 (A. & D. Krepski)

Let $\mathcal{G} \rightrightarrows M$ be equipped with an equivariant prequantization P. Then $\mathrm{Pois}_\bullet(\mathcal{G})$, along with the quantization map Q, define an invertible generalized morphism (i.e. a quasi-isomorphism),

$$\mathrm{Pois}_\bullet(\mathcal{G}) : \mathcal{O}_\bullet \dashrightarrow \mathcal{Q}_\bullet$$

This mirrors Kostant and Souriau's quantization isomorphism in the classical setting.

Kostant-Souriau Central Extension II

Corollary 16 (A. & D. Krepski)

Let M be connected. The sequence of chain complexes,

$$\mathrm{hfib}(\mathrm{Pois}(\mathcal{G})_\bullet) \dashrightarrow \mathcal{Q}_\bullet \dashrightarrow \mathcal{H}_\bullet \dashrightarrow \mathrm{hfib}(\mathrm{Pois}(\mathcal{G})_\bullet)[1] \tag{3}$$

is an exact triangle (i.e. an exact sequence up to homotopy). The induced long exact sequence in homology again yields the classical extension

$$\mathbb{R} \to C^\infty(M) \to \mathrm{Ham}(M)$$

in the case where $\mathcal{G} \rightrightarrows M$ is a manifold.

Conclusion

Conclusion

To summarize the presented results:

- Generalized the Poisson algebra of a (pre-)symplectic manifold to a 0-symplectic groupoid
- Generalized the notion of horizontal lifts for prequantum bundles to groupoids
- Generalized the Kostant-Souriau central extension results to 0-symplectic groupoids

Other things we hope to include in this work:

- Complex line bundle analogues of our results
- Discussion of 2-representations of the algebra
- Obstruction theory of equivariant prequantum bundles via holonomy and cohomology
- Morita invariance

In the future, there are many directions to generalize (higher groupoids, actions of groupoids, symplectic reduction,...)

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