Fedosov Quantization on Smooth Manifolds and Hidden Symmetries

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What's known.

- Kontsevich (2003): star products exist on any Poisson manifold — but hard to compute
- Fedosov (1994): explicit construction via Weyl bundles — but only for *symplectic* manifolds
- Esposito–Nest–Schnitzer–Tsygan (2025): star products admitting quantum moment maps exist via $\mathfrak{g}$-adapted formality — requires equivariant projectivity

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Advantage: Fedosov quantization of X gives us explicit formulas, and symmetries transfer nicely.

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Local picture. $M = \{x \in X \mid x^\alpha = 0\}$ with coordinates (x^i, x^α) on X, where x^i are tangential and x^α are normal to M.

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$$\pi^{ij} = \omega^{j\alpha}\Gamma^i_\alpha - \omega^{i\alpha}\Gamma^j_\alpha, \qquad \mathrm{Rank}(\omega^{i\alpha}) = n$$

with $\omega^{ij} = \omega^{\alpha\beta} = 0$ (bi-Lagrangian condition).

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Flat normal connection.

$$D = dx^\alpha \otimes \left(\partial_\alpha + \Gamma^i_\alpha\partial_i + x^\beta A^i_{\alpha\beta}\partial_i + \cdots\right)$$

where higher coefficients are determined iteratively by $D^2 = 0$.

Lifting Functions from M to X

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The lifting map $\sigma^{-1} : C^\infty(M)[[\hbar]] \to C^\infty(X)[[\hbar]]$: given $f(x^i)$, solve $D(F) = 0$ with

$$F(x^i, x^\alpha) = f + f_\alpha x^\alpha + f_{\alpha\beta} x^\alpha x^\beta + \cdots$$

by determining each $f_{\alpha\beta\ldots}$ iteratively. The inverse σ is restriction to $x^\alpha = 0$.

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Poisson compatibility

$$\{f, g\}_M = \{F, G\}\big|_M$$

The Poisson bracket of M is the *restriction* of the Poisson bracket of (X, ω). This is what makes the entire descent work.

Fedosov Quantization on X and the Normal–Tangential Split

The Weyl bundle $\mathcal{W}$ over X: sections are formal power series in fiber variables (y^I) with Moyal product

$$a \circ b = \exp\left(-\frac{i\hbar}{2}\omega^{IJ}\frac{\partial}{\partial y^I}\frac{\partial}{\partial z^J}\right) a(y)b(z)\Big|_{z=y}$$

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Lemma

- $D^{\perp\,2} = 0$ and $D^\perp$ is a derivation of $\circ$
- $\mathcal{W}^\perp := \ker D^\perp$ is a **subalgebra** of the Weyl algebra

This is the subalgebra that is analogous to $\ker(D)$

The Algebra Isomorphism Φ

Let $\mathcal{W}_M = (C^\infty(M)[[\hbar]])[[y^i]]$ be the Weyl bundle restricted to M.

Define $\Phi : \Gamma(\mathcal{W}_M) \to \Gamma(\mathcal{W}^\perp)$ by: $\Phi(s)$ is the unique $D^\perp$-flat section restricting to s on $\mathcal{W}_M$.

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Theorem (Algebra isomorphism)

Φ is an algebra isomorphism with inverse given by restriction to $y^\alpha, x^\alpha = 0$. The induced product on $\Gamma(\mathcal{W}_M)$ is

$$s_1 \circ_M s_2 = (\tilde{s}_1 \circ \tilde{s}_2)\big|_{y^\alpha, x^\alpha = 0}$$

Proof sketch. Adapted from Fedosov's original argument: write $D^\perp \tilde{s} = 0$ as a recursive equation $\tilde{s} = s_{00} + \delta_n^{-1}(A(\tilde{s}))$, where δ_n^{-1} (the Koszul differential in $D^\perp$) raises y^α-degree. Solve iteratively, uniqueness follows since δ_n^{-1} is strictly degree-raising.

The Star Product on M

Descend everything to M. Define $\bar{D} := \Phi^{-1} D^{\|} \Phi$ on $\mathcal{W}_M$. This is flat by construction. Let $\bar{\sigma}^{-1} : C^\infty(M)[[\hbar]] \to \Gamma(W_M)$ be the Fedosov lifting for $\bar{D}$.

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Theorem (Star product on a Poisson manifold)

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Classical limit. The chain $\Phi \circ \bar{\sigma}^{-1}$ maps into $\ker D_{\mathrm{Fed}}$, so

$$\boxed{f \star_M g = (F \star_X G)\big|_M}$$

where F, G are the Fedosov lifts to X. This immediately gives

$$f \star_M g - g \star_M f = i\hbar\{f, g\}_M + O(\hbar^2)$$

as a restriction of the corresponding identity on (X, ω).

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Proposition (G-invariance of $\star_M$)

Let (M, π) have a Hamiltonian G-action and a G-invariant connection. Then for any G-invariant Fedosov class Ω:

$$\varphi_g^*(f \star_M h) = \varphi_g^* f \star_M \varphi_g^* h \qquad \forall g \in G$$

Note. We only need a G-**invariant connection** on M. Compare with ENST, who require equivariant projectivity of the $\mathfrak{g}$-adapted formality morphism — a stronger condition.

Lifting the Moment Map and the Quantum Defect

Lift. For each $\xi \in \mathfrak{g}$, define $J(\xi) \in C^\infty(X)[[\hbar]]$ by
$\sigma_{\mathrm{Fed}}^{-1}(J(\xi)) = \Phi(\bar{\sigma}^{-1}(\rho(\xi))) \in \ker D_{\mathrm{Fed}} \cap W^\perp$.
Then $J(\xi)\big|_M = \rho(\xi)$ and J is a quantum Hamiltonian: $\frac{1}{i\hbar}[J(\xi), F]_{\star_X} = -\mathcal{L}_{X_\xi} F$.

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The defect from being a Lie algebra homomorphism:

$$\lambda(\xi, \eta) := \frac{1}{i\hbar}[J(\xi), J(\eta)]_{\star_X} - J([\xi, \eta])$$

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Properties [Müller-Bahns-Neumaier, 2004]

- $\lambda(\xi, \eta) \in C[[\hbar]]$ (*constant* on X)
- $\lambda(\xi, \eta) = \Omega(X_\xi, X_\eta)$ where Ω is the Fedosov central curvature
- $[\lambda] \in H^2_{\mathrm{CE}}(\mathfrak{g}, \mathbb{C})[[\hbar]]$ is the obstruction to quantum moment maps

Main Result: Quantum Moment Maps on Poisson Manifolds

Theorem A ($\Omega = 0$, any $\mathfrak{g}$)

The classical moment map ρ is already quantum: $\frac{1}{i\hbar}[\rho(\xi), \rho(\eta)]_{\star_M} = \rho([\xi, \eta])$

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Theorem B ($\mathfrak{g}$ semisimple, any G-invariant Ω)

$\hat{\rho}(\xi) = \rho(\xi) - a(\xi)$ with $a \in \mathfrak{g}^* \otimes \mathbb{C}[[\hbar]]$ uniquely determined by Whitehead homotopy:

$$a(\xi) = -\sum_k \Omega(X_{\zeta^{(k)}}, X_{\eta^{(k)}}), \quad \xi = \sum_k [\zeta^{(k)}, \eta^{(k)}]$$

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Uniqueness: $H^1(\mathfrak{g}, \mathbb{C}) = 0$ (first Whitehead lemma) $\implies$ a is unique.
General $\mathfrak{g}$: obstruction is $[\lambda] \in H^2(\mathfrak{g}, \mathbb{C})[[\hbar]]$, vanishes iff $\Omega(X_\xi, X_\eta) = \delta a$.

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Converse construction. Given a bi-Lagrangian foliated symplectic manifold (X, ω) with a G-action and a Lagrangian leaf M:

- The restriction of ω *induces* a Poisson structure π on M
- The Fedosov star product on X descends to a deformation of (M, π)
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Question. Can one systematically produce non-trivial Poisson structures and their deformation quantizations by choosing Lagrangian embeddings into *explicit* symplectic manifolds?

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3. Connecting two quantizations.

- Fedosov on M (this paper): controlled by $\Omega \in H^2_{\mathrm{dR}}$, explicit star product
- Berezin via RKHS pullback: controlled by m (the dimension), inherits coherent-state structure
- **Goal:** Identify an equivalence relating the two, potentially linking deformation quantization with geometric quantization on Poisson manifolds

Thank You!