Geometric quantization on CR manifolds

- Main motivation (Weyl quantization)

$f \in \mathcal{S}(T^*\mathbb{R}^n)$ we define $f^W : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$

$$\left(f^W \varphi\right)(x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} e^{\frac{i(x-y)\cdot\xi}{\hbar}} f\left(\frac{x+y}{2}, \xi\right) \varphi(y)\, dy\, d\xi$$

- Then, $f, g \in \mathcal{S}(\mathbb{R}^{2n})$

$$[f^W, g^W] = \hbar/i \, \{f, g\}^W + O(\hbar^2)$$

where

$$\{f, g\} = \sum_{k=1}^{n} \left(\frac{\partial f}{\partial \xi_k} \frac{\partial g}{\partial x_k} - \frac{\partial f}{\partial x_k} \frac{\partial g}{\partial \xi_k} \right)$$

Quantization of Hodge manifolds

(M, ω, J) is a Kähler manifold (compact, connected)

$$[\omega] \in H^2(M, \mathbb{Z})$$

There exists a Hermitian complex line bundle (L, h^L) such that the curvature of the unique holomorphic connection compatible with h^L is $-2i\omega$.

$$\mathcal{Q}(M) = \left(\bigoplus_k H^0(M, L^{\otimes k}), \langle \cdot, \cdot \rangle_k \right)$$

For each smooth function f:

$$T_f^{(k)} := B_k \circ M_f \circ B_k \quad , \quad B_k : L^2(M, L^{\otimes k}) \longrightarrow H^0(M, L^{\otimes k})$$

Abstract definition of quantization

A star product $*$ for $(C^\infty(M), \{\ ,\ \})$ is an associative product, $f, g \in C^\infty(M)$

$$f * g = \sum_{J=0}^{+\infty} \hbar^J C_J(f,g) \in C^\infty(M)[[\hbar]] \quad C_J(f,g) \in C^\infty(M)$$

with $C_0(f,g) = f \cdot g$ $\quad C_1(f,g) - C_1(g,f) = i\{f,g\}$

Theorem [Bordemann-Meinrenken-Schlichenmaier]

There exists a unique star product $*_T$ such that

$\forall f, g \in C^\infty(M, \mathbb{C})$, $\forall N \in \mathbb{N}$, $\exists K_N(f,g) > 0$

$$\left\| T_{f,k} T_{g,k} - \sum_{0 \leq J < N} k^{-J} T_{C_J(f,g),k} \right\| \leq K_N(f,g)\, k^{-N}$$

CR manifolds

$(X, T^{(1,0)}X)$ $(2n+1)$-dimensional compact and orientable CR manifold if and only if

- $T^{(1,0)}X \subseteq TX \otimes \mathbb{C}$, $\mathrm{rk}(T^{(1,0)}X) = n$, $T^{(1,0)}X \cap T^{(0,1)}X = \{\underline{0}\}$
- $[C^\infty(X, T^{(1,0)}X), C^\infty(X, T^{(1,0)}X)] \subset C^\infty(X, T^{(1,0)}X)$

Horizontal bundle: $HX \otimes \mathbb{C} = T^{(1,0)}X \oplus T^{(0,1)}X$

Contact form $\omega_0 \in C^\infty(X, T^*X)$, $\langle \omega_0(x), u \rangle = 0 \quad u \in H_x X$

Assume: $U, V \in T_x^{1,0}X \quad L_x(U, \bar{V}) := -\frac{1}{2i} d_x\omega_0(U, \bar{V})$

is non-degenerate. (n_-, n_+)

Operators

$\langle \cdot | \cdot \rangle$ on $TX \otimes \mathbb{C}$ so that $T^{(1,0)}X \perp T^{(0,1)}X$

It induces a metric on $\Lambda^q(T^*X \otimes \mathbb{C})$

$$\pi^{(0,q)} : \Lambda^q(T^*X \otimes \mathbb{C}) \to \Lambda^q(T^{*,(0,1)}X)$$

Definition $\bar{\partial}_b = \pi^{(0,q+1)} \circ d : \Omega^{(0,q)}(X) \to \Omega^{(0,q+1)}(X)$

$$\Box_b^{(q)} := \bar{\partial}_b \bar{\partial}_b^* + \bar{\partial}_b^* \bar{\partial}_b$$

Szegő projector $S^{(q)} : L^2_{(0,q)}(X) \to \operatorname{Ker} \Box_b^{(q)}$

For each $f \in C^\infty(X)$ define

$$T^{(q)}[f] = S^{(q)} \circ M_f \circ S^{(q)}$$

Transversal and CR $\mathbb{R}$-action on X

Let $C^\infty(X)^{\mathbb{R}}$ be the aloebra of $\mathbb{R}$-invariant smoot fuctions

For every $f \in C^\infty(X)^{\mathbb{R}}$ $\exists!\, X_f$ such that

$$\omega_0(X_f) = f \quad \text{and} \quad d\omega_0(X_f, \cdot) = -df$$

For each $f, g \in C^\infty(X)^{\mathbb{R}}$: $\{f, g\} := d\omega_0(X_f, X_g)$

Theorem [JGA, 2023]

The Toeplitz operators $T^{(n)}$'s induce a star star product for the algebra

$$(C^\infty(X)^{\mathbb{R}}, \{\ ,\ \})$$

The Pseudo-Kähler case

The study of Bergmann-Kernel for the mixed curvature case can be found in Ma-Marinescu.

Recall that a pseudo-Kähler manifold (M,ω,J)

- ω is the symplectic structure
- J is the compatible complex structure $(\omega(JX,JY)=\omega(X,Y))$
- $\omega(J\cdot,\cdot)$ has constant signature (n_-,n_+)

Let (L,h^L) be a Hermitian line bundle whose curvature

$$R^L=-2i\omega$$

<u>Definition</u> For each positive integer k, the quantiz. space $Q_k(M)$ is the space of harmonic $(0,n_-)$ forms with values in $L^{\otimes k}$

$$Q_k(M) := \ker \Box_k^{(n_-)}$$

where $\Box_k^{(n_-)}$ is the Kodaira Laplacian acting on $L^{\otimes k}$-valued $(0,n_-)$-forms.

<u>Corollary</u> There exists a unique formal star product

There exists a unique star product $*_T$ such that $\forall f,g \in C^\infty(M,\mathbb{C}), \forall N \in \mathbb{N}, \exists K_N(f,g) > 0$

$$\left\| T_{f,k}^{(n_-)} T_{g,k}^{(n_-)} - \sum_{0 \leq J < N} k^{-J} T_{C_J(f,g),k}^{(n_-)} \right\| \leq K_N(f,g)\, k^{-N}$$