Asymptotics and Dimension Theory of Constrained Quantization for Compactly Supported Measures

ISQGD Special Session 03: Use of Harmonic Analysis in Quantization, Geometry, and Dynamics

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Outline

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Classical Quantization Problem

Quantization is to discretize a probability by finitely supported discrete probability, i.e., the minimization problem for $r \in (0, \infty)$ (Graf & Luschgy, 2007):

$$\inf\{\int \min_{a\in\alpha} \|x-a\|^r dP(x) : \alpha \subseteq \mathbb{R}^D, |\alpha| \leq n\}$$

Alternatively, by optimal transport (Kloeckner, 2012)

$$\inf\{W_r(P,\nu), \nu \text{ is a probability }, |\text{supp}(\nu)| \leq n\}$$

For $r < \infty$ and $r = \infty$, define the *quantization error* (also know as *covering radius* when $r = \infty$) as

$$e_{n,r}(P) := \inf_{\substack{\alpha\subset\mathbb{R}^D \\ |\alpha|\leq n}} \left(\int \min_{a\in\alpha} |x-a|^r \, dP(x)\right)^{1/r}, \; e_{n,\infty}(P) := \inf_{\substack{\alpha\subset\mathbb{R}^D \\ |\alpha|\leq n}} \sup_{x\in\text{supp}(P)} \min_{a\in\alpha} |x-a|.$$

Constrained Quantization

The set α attaining the infimum is called *n-optimal set* or the *n-quantizers*. Consider the problem proposed by Pandey and Roychowdhury, 2026 that

$$\inf\{\int \min_{a\in\alpha} \|x-a\|^r dP(x) : \alpha \subseteq S, |\alpha| \leq n\}$$

where S is a closed set. For $r \in (0,\infty)$ and $r=\infty$,

$$e_{n,r}(P;S) = \inf_{\substack{\alpha\subset S\\ |\alpha|\leq n}} \left(\int \min_{a\in\alpha} |x-a|^r \, dP(x)\right)^{1/r}, e_{n,\infty}(K;S) = \inf_{\substack{\alpha\subset S\\ |\alpha|\leq n}} \sup_{x\in \text{supp}(P)} \min_{a\in\alpha} |x-a|.$$

Errors for Constrained Quantization

Pandey and Roychowdhury, 2026 showed the existence of $n-$quantizers provided $\int |x|^r dP(x) < \infty$ when $r < \infty$, hence $e_{\infty,r}(P;S) = \lim_{n\to\infty} e_{n,r}(P;S)$ is well-defined, and hence we consider the *errors of constrained quantization*,

$$\tilde{e}_{n,r}(P;S) = e_{n,r}(P;S) - e_{\infty,r}(P;S).$$

The well-definedness is guaranteed when $r = \infty$ by an argument through the Hausdorff distance if P is compactly supported.

Dimension Theory for Quantization

In the classical settings, *upper and lower quantization dimensions*

$$\overline{\dim}_Q^r(P;S) = \limsup_{n\to\infty} \frac{\log n}{-\log e_{n,r}(P)} \quad \text{and} \quad \underline{\dim}_Q^r(P) = \liminf_{n\to\infty} \frac{\log n}{-\log e_{n,r}(P)}.$$

are concepts characterizing the decay rate of errors. For the constrained cases, we analogously define

$$\overline{\widetilde{\dim}}_Q^r(P;S) = \limsup_{n\to\infty} \frac{\log n}{-\log \tilde{e}_{n,r}(P;S)} \quad \text{and} \quad \underline{\widetilde{\dim}}_Q^r(P;S) = \liminf_{n\to\infty} \frac{\log n}{-\log \tilde{e}_{n,r}(P;S)}.$$

- Typically, for some 'good' sets with dimension d, we have $e_{n,r}(P) \sim n^{-1/d}$.

Classical Backdrop on Dimension Theory

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- By Graf and Luschgy, 2007, if P is compactly supported and $\mathrm{supp}(P) = K$, then

$$\underline{\dim}_B(K) \geq \underline{\dim}_Q^{\infty}(P) \geq \underline{\dim}_Q^{s}(P) \geq \underline{\dim}_Q^{r}(P),$$
$$\overline{\dim}_B(K) \geq \overline{\dim}_Q^{\infty}(P) \geq \overline{\dim}_Q^{s}(P) \geq \overline{\dim}_Q^{r}(P) \text{ for } 1 \leq r \leq s \leq \infty.$$

- By Graf and Luschgy, 2007, if $r \geq 1$, $\dim_H^*(P) \leq \underline{\dim}_Q^r(P)$ where $\dim_H^*(P) := \inf_{E \subset \mathcal{B}(\mathbb{R}^D), P(E)=1} \dim_H(E)$.
- By Kloeckner, 2012, if P is d-ADR (Ahlfors-David regular) and compactly supported on a stable Riemannian manifold, then $\underline{\dim}_Q^r(P) = \overline{\dim}_Q^r(P) = d$ for $1 \leq r < \infty$.
- By Dai and Liu, 2008, if $\mathrm{supp}(P) \subset \bigtimes_{i=1}^{D} [0, 1]$ and $0 < r < \infty$, then $\dim_H^*(P) \leq \underline{\dim}_Q^r(P)$.

Challenges for Dimension Theory for Constrained Cases

- The positioning of S and K with respect to each other.
- The geometry of S and K thenselves.

Let $K = \mathrm{supp}(P)$ and let $S \subset \mathbb{R}^D$ be closed. Define the multivalued metric projection

$$\pi_S(x) := \{y \in S : |x - y| = \mathrm{dist}(x, S)\}, \qquad \pi_S(K) := \bigcup_{x \in K} \pi_S(x).$$

For $E \subset \pi_S(K)$, define $\pi_S^{-1}(E) := \{y \in K : \pi_S(y) \cap E \neq \emptyset\}$.
A measurable selector is a map

$$T : K \to S, \qquad T(x) \in \pi_S(x) \quad \forall x \in K.$$

which exists by the *Kuratowski–Ryll-Nardzewski measurable selection theorem.*

Upper Asymptotics: Main Results 1

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Theorem (Qian, 2025)

*Suppose $\pi_S(K)$ is **lower** Ahlfors regular of dimension d. Suppose the following condition holds*

$$\text{for } \alpha_n \text{ the } n\text{−quantizer of } e_{n,\infty}(P;S), \quad \lim_{n\to\infty} d_H(\alpha_n, \pi_S(K)) = 0 \tag{U}$$

$$\limsup_{n\to\infty} n\tilde{e}^d_{n,r}(P;S) \leq \limsup_{n\to\infty} n\tilde{e}^d_{n,\infty}(P;S) < \infty,$$

which implies that for $0 < r < \infty$,

$$\dim_H(\pi_S(K)) \geq \overline{\widetilde{\dim}}^r_Q(P;S) \geq \underline{\widetilde{\dim}}^r_Q(P;S),$$

$$\dim_H(\pi_S(K)) \geq \overline{\widetilde{\dim}}^\infty_Q(P;S) \geq \underline{\widetilde{\dim}}^\infty_Q(P;S).$$

- Upper asymptotics are controlled by the geometry of $\pi_S(K)$ and by the selector pushforward measure $T_\#P$.

Upper Asymptotics: Main Results 2

Theorem (Qian, 2025)

Suppose the condition (U) holds,

- *let T be the measurable selector, for $r \geq 1$,*

$$\underline{\widetilde{\dim}}^{r}_{Q}(P;S) \leq \underline{\dim}^{*}_{B}(T_{*}P) \leq \underline{\dim}_{B}(\pi_S(K)),\ \ \overline{\widetilde{\dim}}^{r}_{Q}(P;S) \leq \overline{\dim}^{*}_{B}(T_{*}P) \leq \overline{\dim}_{B}(\pi_S(K));$$

- *for $r > 0$, if $\pi_S(K)$ is **lower** Ahlfors regular of dimension d, then,*

$$\dim_H(\pi_S(K)) \geq \overline{\widetilde{\dim}}^{r}_{Q}(P;S) \geq \underline{\widetilde{\dim}}^{r}_{Q}(P;S), \dim_H(\pi_S(K)) \geq \overline{\widetilde{\dim}}^{\infty}_{Q}(P;S) \geq \underline{\widetilde{\dim}}^{\infty}_{Q}(P;S).$$

Upper Asymptotics: Proof Mechanisms 1

$$\text{for } \alpha_n \text{ the } n\text{—quantizer of } e_{n,\infty}(P;S), \quad \lim_{n\to\infty} d_H(\alpha_n, \pi_S(K)) = 0 \tag{U}$$

Step 1: Pull-back the errors to the covering radius of a compact set

Lemma (Qian, 2025)

- (U) *implies that* $d_H(\pi_S(K), K) \leq \lim_{n\to\infty} \inf_{\substack{\alpha\subseteq S \\ |\alpha|\leq n}} d_H(\alpha, K) = e_{\infty,\infty}(K;S)$
- $e_{n,\infty}(K;S) - e_{\infty,\infty}(K;S) \leq e_{n,\infty}(\pi_S(K);\pi_S(K))$

Upper Asymptotics: Proof Mechanisms 2

$$\text{for } \alpha_n \text{ the } n\text{—quantizer of } e_{n,\infty}(P;S), \quad \lim_{n\to\infty} d_H(\alpha_n, \pi_S(K)) = 0 \tag{U}$$

Step 2: Use a stopping time argument to show that for a compact set $K \subset \mathbb{R}^D$, $ne_{n,\infty}(K) < C(D)$ uniformly on n.

Step 3: Use the fact that T is measurable and push the upper asymptotics to the case that the constraint is $\pi_S(K)$, then use the monotonicity of $e_{n,r}(T_*P; \pi_S(K))$ in (Qian, 2025) and the proof of (Graf & Luschgy, 2007, p. 161).

Lower Asymptotics: A good-λ lower bound technique 1

Suppose $\alpha_n = \{a_i\}_{i=1}^{|\alpha_n|} \subset S$ be an n-optimal constrained quantizer, and let $\{V_i\}_{i=1}^{|\alpha_n|}$ be the associated *Voronoi partition*, and T the measurable selector. For each i, define

$$H_{a_i}(x) := \|x - a_i\| - \|x - T(x)\|, \qquad G_{a_i}(x) := \|a_i - T(x)\|,$$

Notice that $\tilde{e}_{n,1}(P;S) = \sum_{i=1}^{n} \int_{V_i} H_{a_i} dP$. We denote the "good" set, for $\lambda > 0$,
$A^t_\lambda(a_i) = \{x \in K : H_{a_i}(x) \geq \lambda(t + G_{a_i}(x))\}$. and we have a *good-lambda* type estimate

$$P(E \cap \{g_y > \tau\}) \leq P(E \cap \{H_y > \lambda(\tau + t)\}) + P(E \setminus A^t_\lambda(y)) \tag{GL}$$

for all $E \in \mathcal{B}(K)$, $y \in S$, $t > 0$, $\delta \in (0,1)$ and $\lambda > 0$ By (GL), if $F^t_\lambda(a_i) = V_i \cap A^t_\lambda(a_i)$ then

$$P(F^t_\lambda(a_i) \cap \{G_{a_i} > \tau\}) \leq P(F^t_\lambda(a_i) \cap \{H_{a_i} > \lambda(\tau + t)\}) \tag{GL'}$$

Lower Asymptotics: A good-λ lower bound technique 2

$$\int_{F_\lambda^t(a_i)} G_{a_i}\, dP = \int_0^\infty P(F_\lambda^t(a_i)\cap\{G_{a_i}>\tau\})d\tau \overset{\text{(GL')}}{\leq} \int_0^\infty P(F_\lambda^t(a_i)\cap\{H_{a_i}>\lambda(\tau+t)\})d\tau$$

$$\leq \frac{1}{\lambda}\int_0^\infty P(F_\lambda^t(a_i)\cap\{H_{a_i}\}) = \frac{1}{\lambda}\int_{F_\lambda^t(a_i)} H_{a_i}\, dP = \tilde{e}_{n,1}(P;S)$$

On the other hand by Hölder's inequality and the Frostman's lemma,

$$\sum_{i=1}^{|\alpha_n|}\int_{F_\lambda^t(a_i)} dP = \sum_{i=1}^{|\alpha_n|}\int_{F_\lambda^t(a_i)} |a_i - T(x)|dP(x) = \sum_{i=1}^{|\alpha_n|}\int_{T(F_\lambda^t(a_i))} |a_i - y| d(T_*(P|_{F_\lambda^t(a_i)})(y))$$

$$\geq c\sum_{i=1}^{|\alpha_n|} cT_*(P|_{F_\lambda^t(a_i)})(T(G_i))^{1+1/d} \geq c\sum_{i=1}^{|\alpha_n|} P(F_\lambda^t(a_i))^{1+1/d} \geq cn^{-1/d}(\sum_{i=1}^{|\alpha_n|} P(F_\lambda^t(a_i)))^{1+1/d}$$

Consider $\eta_n(\lambda,t) = \sup\{\sum^{|\alpha_n|} P(V_i\cap A_\lambda^t(a_i)) : \{a_i\}_i = \alpha \text{ is } n\text{−quantizer}\}$

Lower Asymptotics: Main Results

- When can $\eta_n(\lambda, t)$ exists some uniform lower bound on n fixing λ, t?

Proposition (Qian, 2025)

Assuming the following condition

$$\begin{aligned}&\text{there exists a sequence of } n\text{-optimal quantizers } \alpha_n \text{ for } e_{n,r}(P;S) \text{ such that:}\\ &\text{for every } \varepsilon > 0,\ \text{every } n \in \mathbb{N},\ \text{and every } a \in \alpha_n \cap \pi_S(K),\\ &\text{there exists } \tilde{a} \in S \setminus \pi_S(K) \text{ such that } \rho(a, \tilde{a}) < \frac{\varepsilon}{n^{1/d}}\end{aligned} \tag{L}$$

and in addition assume that $T_* P$ *is* ***upper*** *Ahlfors regular, then* $\underline{\dim}_Q^r(P; S) \geq \dim_H^*(T_* P)$ *and* $\underline{\dim}_Q^r(P; S) \geq \dim_H(\mathrm{supp}(T_* P))$

Application: Sharp Dimension Equality

Corollary (Qian, 2025)

Suppose the following condition

$$\exists C > 0, \quad \forall \varepsilon > 0, \quad \forall x \in \pi_S(K), \quad P(\pi_S^{-1}(B(x,\varepsilon)\cap S)) > C\varepsilon^s \text{ where } s = \dim_H(\pi_S(K)) \tag{U'}$$

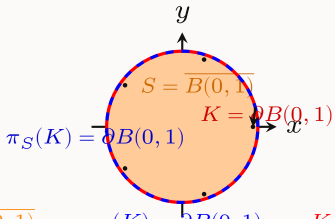
and the condition (L) hold. In addition suppose T is surjective and T_*P is ADR, then

$$\dim_H(\pi_S(K)) = \underline{\dim}_Q^r((P;S)) = \underline{\dim}_Q^\infty((P;S)) = \overline{\dim}_Q^r((P;S)) = \overline{\dim}_Q^\infty((P;S))$$

For the lower bound

Example: Quantizers outside $\pi_S(K)$

$S = \overline{B(0,1)}$ and $K = \partial B(0,1)$ in $\mathbb{R}^2$, then the measurable selector $T : K \to S, x \mapsto x$, is the identity map. Let P be uniform on $\partial B(0,1)$. $\pi_S(K) = K$ has dimension of 1 in the sense of Hausdorff, box, and Ahlfors and quantization. $\overline{\widetilde{\dim}}_Q^r(P; S) = 1$ for $r \geq 1$. The n-quantizers form a regular n–gon $\alpha_n = \{(\theta_i, \frac{n}{\pi}\sin\left(\frac{\pi}{n}\right)) : \theta_i = \frac{2\pi i}{n},\ 0 \leq i \leq n-1\}$, which coincide with (Rosenblatt & Roychowdhury, 2023). Notice there is a sequence of $n-$quantizers strictly inside the unit disc. Hence $\underline{\widetilde{\dim}}_Q^r(P; S) = 1$.



$S = \overline{B(0,1)}$ • $\pi_S(K) = \partial B(0,1)$ • $K = \partial B(0,1)$
P: uniform distribution on K • α_5: optimal 5-point constrained quantizer

Remarks on Further Directions

- Can we construct dimension gaps via $\eta_n(\lambda, t)$ besides examples by Pandey and Roychowdhury, 2024?
- In the sharp equality, in what cases does $\dim_B(\pi_S(K))$ equal to $\dim_H(\pi_S(K)) = \underline{\dim}_Q^r((P;S)) = \underline{\dim}_Q^\infty((P;S)) = \overline{\dim}_Q^r((P;S)) = \overline{\dim}_Q^\infty((P;S))$?
- Besides ADR, can we use UR/NTA to control the geometry of $\pi_S(K)$ and use Carleson/conical condition to control the density of P to get sharp asymptotics?

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