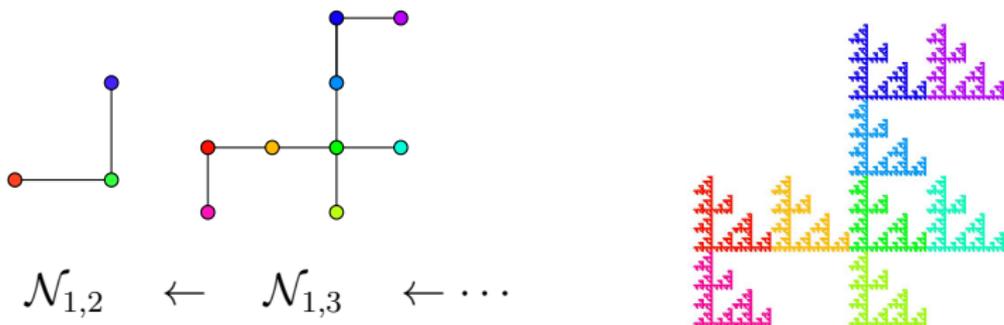


On the topology of the limit set of non-autonomous IFS

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Introduction to non-autonomous IFS

Simplicial method for non-autonomous IFS

Hata's Theorem and more

Homology groups of non-autonomous fractal rectangle

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Non-autonomous IFS

Definition 1

- ▶ Let X be a compact metric space.
- ▶ Let $\Phi^{(j)} = \{f_i^{(j)} : X \rightarrow X\}_{i \in I^{(j)}}$ be a collection of maps with $\#I^{(j)} < \infty$ for every $j \geq 1$.
- ▶ Denote the Lipschitz constant by $\text{Lip}(f)$ for a map f .
- ▶ Assume that $\exists c < 1$ such that $\text{Lip}(f_i^{(j)}) \leq c$ for each $j \geq 1$ and for each $i \in I^{(j)}$.

We call $(\Phi^{(j)})_{j=1}^{\infty}$ a **non-autonomous iterated function system**.

For $(\Phi^{(j)})_{j=1}^{\infty}$, define **the limit set** by

$$J := \bigcap_{j=1}^{\infty} \bigcup_{(i_1, i_2, \dots, i_j) \in \prod_{\ell=1}^j I^{(\ell)}} f_{i_1}^{(1)} \circ f_{i_2}^{(2)} \circ \dots \circ f_{i_j}^{(j)}(X).$$

We say that $(\Phi^{(j)})_{j=1}^{\infty}$ is autonomous if all $\Phi^{(j)}$'s are identical.

Previous Studies: connectedness of **auto.** limit set

Theorem 2 (Hata)

For the limit set J of an autonomous IFS $\{f_i\}_{i \in I}$, TFAE.

- ▶ J is connected.
- ▶ J is path-connected.
- ▶ For every $i, i' \in I$, there exist $n \in \mathbb{N}$ and $i_0, i_1, \dots, i_n \in I$ such that $i = i_0$, $i' = i_n$, and $f_{i_k}(J) \cap f_{i_{k+1}}(J) \neq \emptyset$ for every $k = 0, \dots, n - 1$.

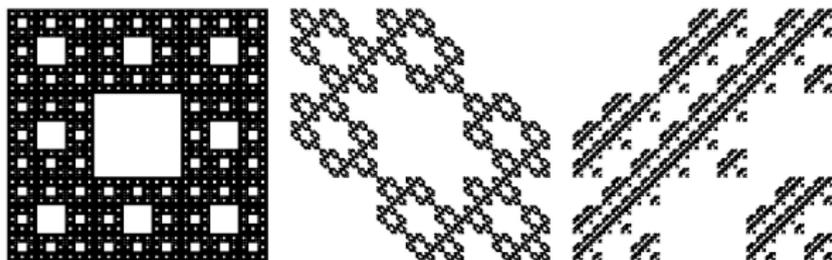


Figure: Sierpiński carpet, (a figure homeo. to) Hexakun, and a disconnected fractal.

Previous Studies: fractal square and randomization

Example 3 (fractal squares)

- ▶ $X = [0, 1]^2$.
- ▶ Let $n \in \mathbb{N}_{\geq 2}$. Define a contractive map $f_{\mathbf{i}} : X \rightarrow X$ by

$$f_{\mathbf{i}}(x_1, x_2) = \left(\frac{x_1 + i_1}{n}, \frac{x_2 + i_2}{n} \right)$$

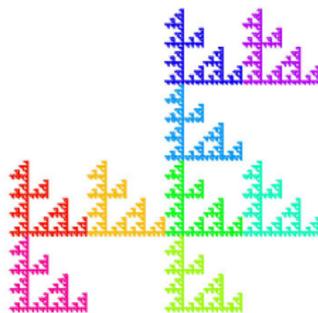
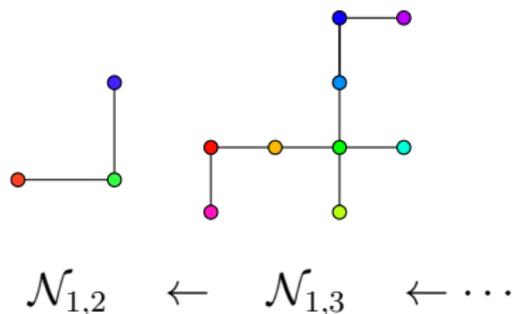
for each $\mathbf{i} = (i_1, i_2) \in \{0, 1, \dots, n-1\}^2 =: I$.

For every $j \geq 1$, a subset $\emptyset \neq I^{(j)} \subset I$ is given. Then $\Phi^{(j)} = \{f_{\mathbf{i}}\}_{\mathbf{i} \in I^{(j)}}$ forms a non-autonomous IFS.

- ▶ Roinestad '10 gave a equivalent condition for autonomous J to be totally disconnected.
- ▶ Xiao '21 gave a equivalent condition for autonomous J to have finitely many connected components.
- ▶ Cristea '08 studied the (total dis)connectedness of **randomly generated non-autonomous fractal squares**.

What we shall do

- ▶ Topology of fractal is not fully understood
- ▶ while dimension theory is well developed.
- ▶ We shall generalize Hata's theorem to the non-autonomous setting, and
- ▶ we aim to investigate the **homology groups of fractal squares**.



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Hutchison-type formula

Let $(\Phi^{(k)})_{k=1}^{\infty}$ be a non-autonomous IFS.

Lemma 4

For every $j \geq 1$, define a non-autonomous IFS $(\Phi^{(j-1+k)})_{k=1}^{\infty}$ and denote its limit set by J_j . Then $J = J_1$ and

$$J_j = \bigcup_{(i_j, \dots, i_{k-1}) \in I^{(j)} \times \dots \times I^{(k-1)}} f_{i_j}^{(j)} \circ \dots \circ f_{i_{k-1}}^{(k-1)}(J_k).$$

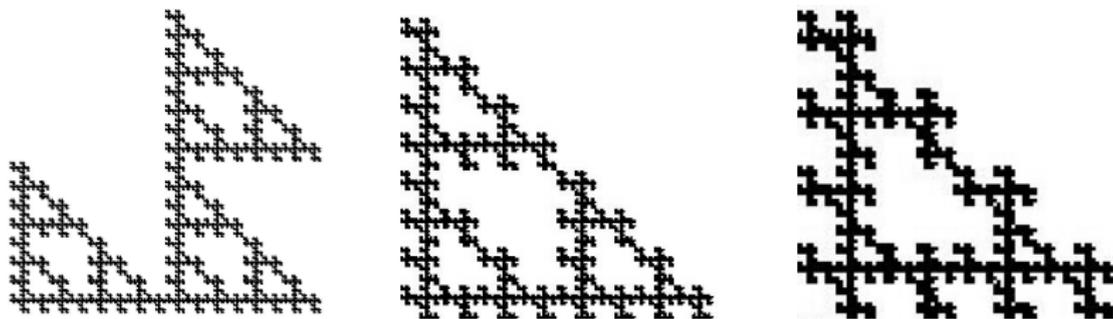
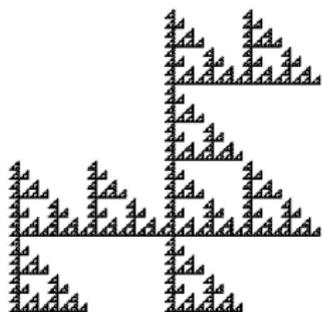


Figure: J_1 , J_2 , and J_3 . Observe the “self-similarity”.

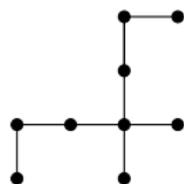
The nerve

Definition 5 (Inspired by Hiroki Sumi '09)

For every $1 \leq j < k$, define the simplicial complex $\mathcal{N}_{j,k}$ as the **nerve** of $\{f_{i_j}^{(j)} \circ \dots \circ f_{i_{k-1}}^{(k-1)}(J_k)\}_{(i_j, \dots, i_{k-1}) \in I^{(j)} \times \dots \times I^{(k-1)}}$ which is a covering of J_j .



$\mathcal{N}_{1,2}$



$\mathcal{N}_{1,3}$

\dots

The inverse limit of homology groups of nerves

Recall that the vertex set is $V(\mathcal{N}_{j,k}) = I^{(j)} \times \cdots \times I^{(k-1)}$.

Definition 6

We define the simplicial map $\phi_{j,k} : \mathcal{N}_{j,k+1} \rightarrow \mathcal{N}_{j,k}$ so that

$$\phi_{j,k}(i_j, \dots, i_{k-1}, i_k) = (i_j, \dots, i_{k-1}).$$

For every $q \geq 0$, this induces $(\phi_{j,k})_* : H_q(\mathcal{N}_{j,k+1}) \rightarrow H_q(\mathcal{N}_{j,k})$ on the homology groups. Take **the inverse limit** as $k \rightarrow \infty$.

Theorem 7

There is an isomorphism $\varprojlim_k H_q(\mathcal{N}_{1,k}) \cong \check{H}_q(J)$ for $\forall q \geq 0$.

Here, $\check{H}_q(J)$ is the Čech homology group, the inverse limit of nerve $N(\mathcal{U})$ of finite open coverings \mathcal{U} . Dually, there is an isomorphism $\varinjlim_k H^*(\mathcal{N}_{1,k}) \cong \check{H}^*(J)$.

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Main theorem on connectedness

Definition 8

For a simplicial complex \mathcal{K} , we denote by $\text{Con}(\mathcal{K})$ the set of all components of \mathcal{K} .

Definition 9

For a top. sp. Y , we denote by $\text{Con}(Y)$, respectively $\text{pCon}(Y)$, the set of all (resp. path-)connected components of Y .

Theorem 10

A map $\text{Con}(J) \ni C \mapsto \mathcal{C}_k \in \text{Con}(\mathcal{N}_{1,k})$ can be defined so that

$$C \subset \bigcup_{(i_1, \dots, i_{k-1}) \in V(\mathcal{C}_k)} f_{i_1}^{(1)} \circ \dots \circ f_{i_{k-1}}^{(k-1)}(J_k)$$

for all $k > 1$. It induces a bijection $\text{Con}(J) \rightarrow \varprojlim \text{Con}(\mathcal{N}_{1,k})$.
Respectively, there is a bijection $\text{pCon}(J) \rightarrow \varprojlim \text{Con}(\mathcal{N}_{1,k})$.

Corollary: a Hata-type theorem

Since $\text{Con}(J) \cong \varprojlim \text{Con}(\mathcal{N}_{1,k}) \cong \text{pCon}(J)$, we have

Corollary 11

The following are equivalent.

1. *The limit set J is connected.*
2. *The limit set J is path-connected.*
3. *For every $k > 1$, $\mathcal{N}_{1,k}$ is connected.*

The third is equivalent to the following:

For every $k > 1$, for every $u, v \in \prod_{\ell=1}^{k-1} I^{(\ell)}$, there exist $n \in \mathbb{N}$ and $v_0, v_1, \dots, v_n \in \prod_{\ell=1}^{k-1} I^{(\ell)}$ such that $u = v_0$, $v = v_n$, and

$$f_{v_i}(J_k) \cap f_{v_{i+1}}(J_k) \neq \emptyset$$

for every $i = 0, \dots, n-1$. Here, $f_v = f_{i_1}^{(1)} \circ \dots \circ f_{i_{k-1}}^{(k-1)}$ for $v = (i_1, \dots, i_{k-1})$.

Local connectedness

Corollary 12

If $\mathcal{N}_{k,k+1}$ is connected for every $k \geq 1$, then the limit set J is connected and **locally connected**.

Remark 3.1

If an autonomous limit set is connected, then it is locally connected. However, this does not hold for non-autonomous limit sets.

A sufficient condition to be totally disconnected

Since the map $\text{Con}(J) \ni C \mapsto \mathcal{C}_k \in \text{Con}(\mathcal{N}_{1,k})$ is defined s.t.

$$C \subset \bigcup_{(i_1, \dots, i_{k-1}) \in V(\mathcal{C}_k)} f_{i_1}^{(1)} \circ \dots \circ f_{i_{k-1}}^{(k-1)}(J_k),$$

we have

$$\text{diam}(C) \leq \sum_{(i_1, \dots, i_{k-1}) \in V(\mathcal{C}_k)} \text{diam}\left(f_{i_1}^{(1)} \circ \dots \circ f_{i_{k-1}}^{(k-1)}(J_k)\right).$$

Recall that $c < 1$ is the uniform upper bound of Lip. constants.

Corollary 13

If $\lim_{k \rightarrow \infty} c^k \max\{\#V(\mathcal{K}) : \mathcal{K} \in \text{Con}(\mathcal{N}_{1,k})\} = 0$, then the limit set J is totally disconnected.

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Recall: fractal rectangle

In this section, we consider the following class.

Definition 14 (Non-autonomous fractal rectangle)

- ▶ $X = [0, 1]^2$.
- ▶ Let $n_1, n_2 \in \mathbb{N}_{\geq 2}$. Define a contraction $f_{\mathbf{i}} : X \rightarrow X$ by

$$f_{\mathbf{i}}(x_1, x_2) = \left(\frac{x_1 + i_1}{n_1}, \frac{x_2 + i_2}{n_2} \right)$$

for each $\mathbf{i} = (i_1, i_2) \in \prod_{k=1}^2 \{0, 1, \dots, n_k - 1\} =: I$.

- ▶ Let $r \in \mathbb{N}$ such that $1 \leq r \leq \prod_{k=1}^2 n_k - 1$. We randomly chose each $I^{(j)}$ independently so that $\#(I \setminus I^{(j)}) = r$.

Then $\Phi^{(j)} = \{f_{\mathbf{i}}\}_{\mathbf{i} \in I^{(j)}}$ forms a non-autonomous IFS.

Connectedness

Theorem 15

Suppose that $r < 2$. Then $\check{H}_0(J) = \varprojlim H_0(\mathcal{N}_{1,k}) \cong \mathbb{Z}$, and hence the limit set J is **connected** and locally connected.

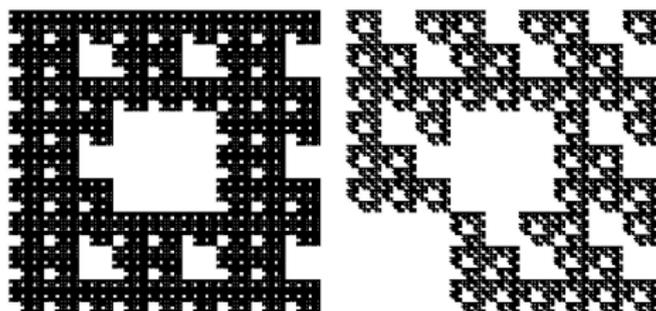


Figure: $n_1 = n_2 = 3$. $r = 1$ (left) and $r = 2$ (right).

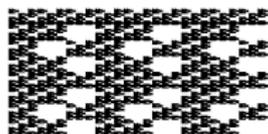
Lemma 16

If $r \geq 2$, then $\text{rank} \check{H}_0(J) = \infty$ almost surely.

Total disconnectedness

Definition 17

Let $I' \subset \prod_{k=1}^2 \{0, 1, \dots, n_k - 1\}$. We say that I' has a **horizontal cut** if $\exists i \in \{0, 1, \dots, n_2 - 1\}$ s.t. $i \notin \pi_2(I')$. Here, π_2 is the projection onto the 2nd coordinate.



Similarly, **vertical cut** is defined.

Corollary 18

If $\#\{j \geq 1 : I^{(j)} \text{ has horizontal cut}\} = \infty$ and
 $\#\{j \geq 1 : I^{(j)} \text{ has vertical cut}\} = \infty$,
then J is **totally disconnected**.

Theorem 19 (Nakajima-W. '25+)

The limit set J satisfies $\check{H}_q(J) = 0$ for every $q \geq 2$. Moreover, almost surely we have the following.

1. If $r = 1$, then $\check{H}_0(J) \cong \mathbb{Z}$.
 - 1.1 If $n_1 = n_2 = 2$, then $\check{H}_1(J) = 0$.
 - 1.2 If $(n_1, n_2) \neq (2, 2)$, then

$$\lim_{k \rightarrow \infty} \frac{1}{k} \log(\text{rank} H_1(\mathcal{N}_{1,k})) = \log(n_1 n_2 - r).$$

2. If $2 \leq r < \min\{n_1, n_2\}$, then

$$\lim_{k \rightarrow \infty} \frac{1}{k} \log(\text{rank} H_1(\mathcal{N}_{1,k})) = \log(n_1 n_2 - r)$$

and $\text{rank} \check{H}_0(J) = \text{rank} \check{H}_1(J) = \infty$.

3. If $n_1 \leq r < n_2$ (resp. $n_2 \leq r < n_1$), then $\check{H}_1(J) = 0$ and $\text{rank} \check{H}_0(J) = \infty$, and J contains a continuum.
4. If $r \geq \max\{n_1, n_2\}$, then J is totally disconnected.

Gallery

Let $n_1 = 3$ and $n_2 = 4$. Remove r rectangles at each step.

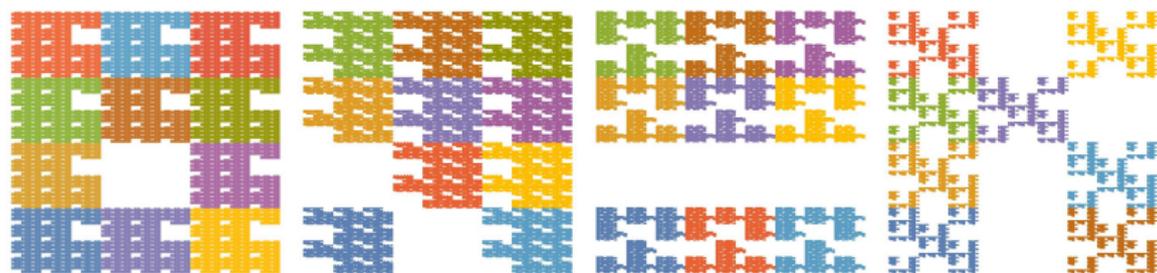


Figure: From left to right, $r = 1, 2, 3, 4$.

Typically, we have

1. $\check{H}_0(J) \cong \mathbb{Z}$ and $\text{rank} \check{H}_1(J) = \infty$.
2. $\text{rank} \check{H}_0(J) = \text{rank} \check{H}_1(J) = \infty$.
3. $\text{rank} \check{H}_0(J) = \infty$ and $\check{H}_1(J) = 0$, but J is not totally disconnected.
4. J is totally disconnected.

Remark

Moreover, each homology group $H_q(\mathcal{N}_{1,k})$ of fractal rectangles is free, and hence the cohomology group $H^q(\mathcal{N}_{1,k})$ is isomorphic to the free abelian group of the same rank.

By the **Alexander duality theorem**

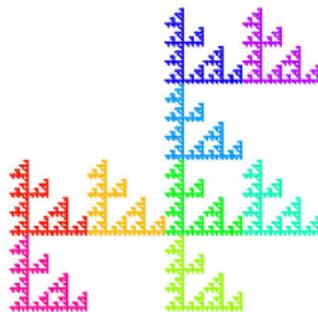
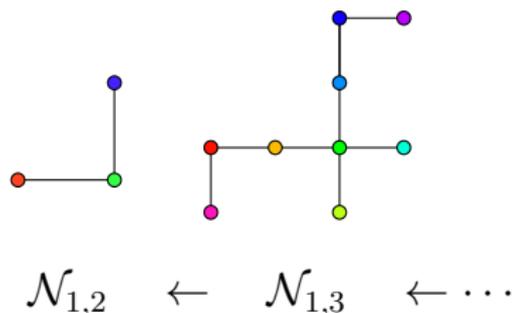
$$\check{H}^{d-q-1}(J; G) \cong \tilde{H}_q(\mathbb{R}^d \setminus J; G),$$

the Čech cohomology of J is related to the reduced homology of its complement.

With $d = 2$ and $q = 0$ we deduce that the number of bounded connected components of $\mathbb{R}^2 \setminus J$ is either zero or infinite.

Summary

- ▶ We defined the simplicial complexes $\mathcal{N}_{j,k}$ using the “self-similar” structure.
- ▶ Our group is isomorphic to the Čech (co)homology group: $\check{H}_*(J) \cong \varprojlim H_*(\mathcal{N}_{1,k})$ and $\check{H}^*(J) \cong \varinjlim H^*(\mathcal{N}_{1,k})$.
- ▶ We established a general theory and studied non-autonomous fractal squares.
- ▶ We can calculate (co)homology groups, and observed that the growth rate of the rank is a meaningful quantity.



Thank you!

