On first order operators on metric graphs and fractals

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ISQGD-SS02 - Fractal Interpolation and Iterated Function Systems, Feb 21, 2026

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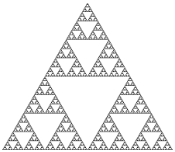
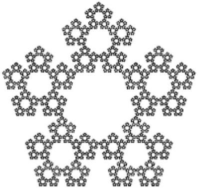
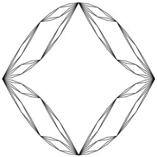
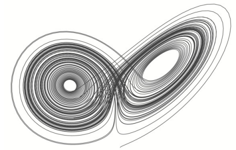
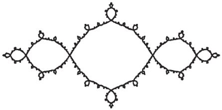
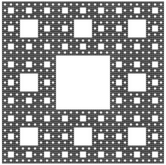
Main topic of talk: analysis on fractal spaces

Motivations:

- Math physics and probability (Mandelbrot percolation, Liouville quantum gravity, random fields, ...)
- Spectral behaviour of hierarchical / dynamical structures (complicated at all scales)
- ”Exotic phenomena” (sub-Gaussian diffusions, singular energy measures, localized eigenfunctions)
- Limits of networks, ”discrete to continuous”

Some industrial applications:

- ”Extreme surface area”: optical waveguides / topological insulators
- ”Multiscale obstacles”: fractal absorbers in acoustics
- ”Self-similar structure”: broadbend antennas (Bluetooth, WLAN, GSM, GPS)

Challenges:

- If embedded in $\mathbb{R}^n$, closed set with empty interior
- No submanifold structure / no smooth charts
- No ”gluing together from smooth pieces” (in contrast to metric graphs or rectifiable sets)
- Too few rectifiable curves
- No curvature(-dimension) bounds
- In general, no Poincaré inequality + doubling measure

Late 1980’s - end of 1990’s: Goldstein, Barlow, Bass, Kusuoka, Kigami, Strichartz, etc.: approach to analysis on fractals using diffusion processes and Dirichlet forms

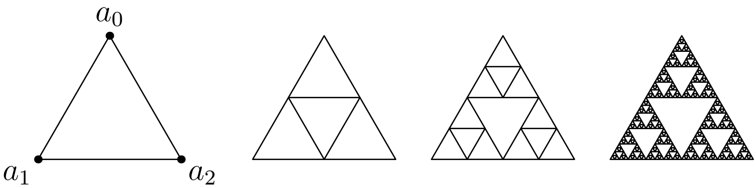
Easiest example: "Standard" energy on SG (Kigami):

$$\mathcal{E}(f) = \text{ "}\int_\Omega |\nabla f(x)|^2 dx\text{ "}$$

as the (rescaled) limit

$$\mathcal{E}(f) := \frac{1}{2}\lim_n \left(\frac{5}{3}\right)^n \sum_{p,q\in V_n,\ q\sim_n p} (f(p)-f(q))^2$$

of discrete energy forms on approximating graphs $G_n = (V_n, E_n)$.



Same as to construct the Brownian motion as the rescaled limit of simple random walks on the graphs $G_n = (V_n, E_n)$.

To find correct rescaling, solve a sequence of discrete Dirichlet problems.

Get a space $\mathcal{F}$ $'' = H^1(\Omega)''$ of functions on SG with finite energy, i.e.

$$\mathcal{E} : \mathcal{F} \to [0, +\infty).$$

Simultaneously get a (resistance) metric ϱ_R on SG so that

$$\mathcal{F} \subset C(SG)$$

(Sobolev embedding theorem).

Construction purely combinatorial, *no volume measure used*.

With "any reasonable" finite Borel measure ν on SG the pair $(\mathcal{E},\mathcal{F})$ becomes a Dirichlet form on $L^2(SG,\nu)$.

Gauss-Green gives Laplacian "$(\Delta,\mathcal{D}_\nu(\Delta))$" for (volume) measure ν,

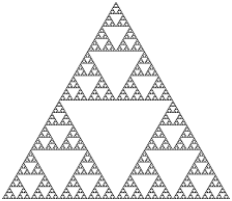
$$\mathcal{E}(f,g)=-\int_{SG} f\Delta g\, d\nu, \qquad f\in\mathcal{F} \quad (\text{resp. "}\mathcal{F}_{V_0}\text{"}).$$

Knowing Δf, can study *second order* PDE

$$\Delta u = f \qquad \text{or} \qquad u_t=\Delta u, \quad u_0=\mathring{u}.$$

"Well understood" by now.

Energy form $\mathcal{E}$ and Brownian motion $(B_t)_{t>0}$ on Sierpinski gasket (SG) ... cells with finite boundaries



Existence and uniqueness: Goldstein 1986, Kusuoka 1986, Barlow / Perkins 1987, Kusuoka 1989, Kigami 1989.

Brownian motion $(B_t)_{t>0}$ on SG with self-similar measure μ:

- $d_H = \frac{\log 3}{\log 2}$ *Hausdorff dimension* of SG,
  $d_w = \frac{\log 5}{\log 2} > 2$ *walk index* (particle moves distance t^{1/d_w} in time t, "slower than normal") ... have

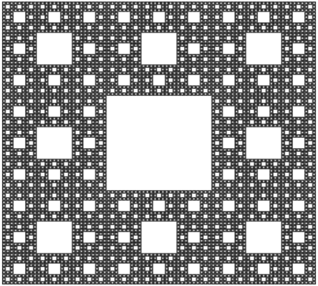
$$d_S = \frac{2d_H}{d_w} < 2 \textit{ spectral dimension}, \text{ short time exponent}$$

- $(B_t)_{t>0}$ is sub-Gaussian, i.e.

$$p(t,x,y) \sim ct^{-d_s/2} \exp\left(-c\left(\frac{\varrho_R(x,y)^{d_w}}{t}\right)^{1/(d_w-1)}\right).$$

- log-scale fluctuations in $t^{d_s/2}p(t,x,x)$ (Kajino 2009)

Energy form $\mathcal{E}$ and Brownian motion $(B_t)_{t>0}$ on Sierpinski carpet (SC) ... cells with uncountable boundaries



Existence: Barlow/Bass 1988, 1998, Uniqueness: Barlow/Bass/Kumagai/Teplyaev 2010.

- *Barlow, 'Diffusions on Fractals', Springer LNM, 1998*
- *Kigami, 'Analysis on Fractals', Cambridge Univ. Press, 2001*
- *Strichartz, 'Introduction to Differential Equations on Fractals', Princeton Univ. Press, 2006*

For self-similar measures μ, *no way to write*

$$\mathcal{E}(f) = \int {}'' \, |\nabla f|^2 \, '' \, d\mu$$

(Ben-Bassat/Strichartz/Teplyaev 1999, Hino 2004, 2006), only $\mathcal{E}(f) = \int d\nu_f$, where

$$d\nu_f = {}'' \, |\nabla f|^2 \, d\mu \; ''$$

Energy and volume are mutually singular,

$$\nu_f \perp \mu$$

Typical for a large class of fractal spaces:

- ”Martingale dimension” (Kusuoka, Hino) $=$ degree of freedom for diffusion $=$ ”a.e. dimension of cotangent spaces” is one

  (although Hausdorff dimension can be arbitrarily large (Laakso 2000))

- Until recently, no robust notion of differential *df* or gradient ∇f or first order derivative on fractals
- Concepts *did exist in other areas*: Mokobodzki, LeJan 1978, Ikeda/Manabe 1976, Nakao 1985, Weaver 2000, Cipriani/Sauvageot 2003
- *df* on fractals: Cipriani/Sauvageot 2009, Ionescu/Rogers/Teplyaev 2012, H./Röckner/Teplyaev 2013, etc.

Take "universal derivations" $\delta_0 f(x,y) = f(x) - f(y)$, use Hilbert seminorm

$$\|\overline{g}\delta_0 f\|_{\mathcal{H}}^2 := \frac{1}{2}\lim_n \left(\frac{5}{3}\right)^n \sum_{p,q\in V_n,\ q\sim_n p} |(\overline{g}\delta_0 f)(p,q)|^2,$$

on span of all $(\overline{g}\delta_0 f)(p,q) = \frac{1}{2}(g(p)+g(q))\delta_0 f(p,q)$ with $f, g \in \mathcal{F}$

Factoring out zero seminorm elements and completing, obtain a Hilbert space $\mathcal{H} = \text{''}L^2(\Omega, \mathbb{R}^n)\text{''}$ of "L^2-vector fields" and a first order derivation operator ("gradient / exterior derivative")

$$\partial : \mathcal{F} \to \mathcal{H},$$

∂f defined as the is $\mathcal{H}$-class of $\delta_0 f$.

Have $\partial(fg) = f\partial g + g\partial f$ and

$$\|\partial f\|_{\mathcal{H}}^2 = \mathcal{E}(f) = \text{''} \int |\nabla f|^2 \, dx \text{''}, \quad f \in \mathcal{F}.$$

Examples

On $X = [0,1]$ or $X = S^1$ have $\partial f = df = f' dx$ (exterior derivative)

First order equations on fractals: hardly anything known

- H./Teplyaev 2015: reduction of Navier-Stokes system to Euler system (vector valued) with stationary solutions
- H./Schefer 2024: first results for equations of continuity and transport type (scalar valued)

$$\begin{cases} \partial_t u + \operatorname{div}(ub) = 0 & \text{in } (0,T) \times X, \\ u(0) = \mathring{u} & \text{on } X \end{cases} \tag{CE}$$

$$\begin{cases} \partial_t w + b \cdot \nabla w = 0 & \text{in } (0,T) \times X, \\ w(0) = \mathring{w} & \text{on } X \end{cases} \tag{TE}$$

Equations coincide for stationary divergence free b.

- $X = \mathbb{R}^n$ and b C^1-vector field: well-posedness via Cauchy-Lipschitz
- $X = \mathbb{R}^n$ and b Sobolev or BV: weak well-posedness via vanishing viscosity and commutation (DiPerna/Lions 1989, Ambrosio 2004), similarly for X metric measure space with curvature bounds (Ambrosio/Trevisan 2014)
- X Hilbert space: well-posedness for measure valued solutions (e.g. Bogachev/DaPrato/Röckner/Shaposhnikov 2015)
- X metric graph: well-posedness via semigroups, ”solve on edges and glue” (Kramar/Sikolya 2005, ...)

All fail for fractals, well-posedness remained open for many years.

Comment on first results, joint with W. Schefer, Bielefeld (PhD 2024)

In sequel, assume that X compact metric space, $(\mathcal{E},\mathcal{F})$ ”local regular energy form” on X

Examples

- $X=[0,1]$, $\mathcal{E}(f)=\int_0^1(f'(x))^2dx$, $f\in H^1(0,1)$
- $X=SG$, $(\mathcal{E},\mathcal{F})$ standard energy form

- Can construct $\mathcal{H}$ and ∂, *no volume measure needed*
- Can introduce energy measures ν_f and ν_ω of functions $f\in\mathcal{F}$ and elements ω of $\mathcal{H}$ (below)
- Call $f\in\mathcal{F}$ resp. ν_f ”minimal energy dominant (m.e.d.)” if ”$\nexists$ nonempty open set on which ∂f vanishes”, and minimal in terms of abs. cont. (Hino 2010), sim. for $\omega\in\mathcal{H}$ and ν_ω

Key tool: Poincaré duality (Baudoin/Kelleher 2016)

- Each $\omega \in \mathcal{H}$ "$= L^2(\Omega, \mathbb{R}^n)$" has an energy measure ν_ω,

$$\langle g\omega, \omega\rangle_{\mathcal{H}} = \int_X g\, d\nu_\omega, \quad g \in C(X)$$

- map $f \mapsto \star_\omega f := f\omega$ injective and an isometry $\star_\omega : L^2(X, \nu_\omega) \to \mathcal{H}$ ("Hodge star operator")
- $\star_\omega$ surjective $\Leftrightarrow$ $(\mathcal{E}, \mathcal{F})$ has martingale dim one and ω is m.e.d. (Baudoin/Kelleher 2016) ... no exotic property (can be relaxed), inverse is $\star_\omega^{-1}(f\omega) = f$

Examples

For $X = [0, 1]$ with $\omega = dx$ find $f \mapsto \star f := fdx$ (classical).

In sequel, assume that martingale dim. is one, and that and ω is a m.e.d. element of $\mathcal{H}$

- Consider ∂ as unbounded lin. op. from $L^2(X,\nu_\omega)$ into $\mathcal{H}$ with dense domain $\mathcal{F}$
- Adjoint ∂^* " = *div* " of ∂ is

$$\mathcal{D}_{\nu_\omega}(\partial^*) = \{v \in \mathcal{H} : \exists g \in L^2(X,\nu_\omega) : \\ \langle v, \partial f\rangle_{\mathcal{H}} = \langle g, f\rangle_{L^2(X,\nu_\omega)}\,,\ f \in \mathcal{F}\}$$

and $\partial^* v := g$. Unbounded lin. op. from $\mathcal{H}$ into $L^2(X,\nu_\omega)$

- Define *function-valued* "differential operators"

$$\star_\omega^{-1}\partial : \mathcal{F} \to L^2(X,\nu_\omega)$$

and

$$\partial^\perp := -\partial^* \star_\omega : \mathcal{D}_{\nu_\omega}(\partial^\perp) \to L^2(X,\nu_\omega),$$

where $\mathcal{D}_{\nu_\omega}(\partial^\perp) := \star_\omega^{-1}\mathcal{D}(\partial^*) = \{f \in L^2(X,\nu_\omega) : \star_\omega f \in \mathcal{D}_{\nu_\omega}(\partial^*)\}$

- Special cases of $\star_\omega^{-1}\partial$ already studied by Hino 2010
- Also consider variants based on

$$\mathcal{F}_B := \{f \in \mathcal{F} : f(p) = 0,\ p \in B\},$$

where $B \subset X$ closed and ν_ω-null, write ∂_B etc. (boundary)
Talk: Mostly $B = \emptyset$, comment on case of nonempty B

Examples

On $X = [0,1]$ with $\omega = dx$ and $B = \{0,1\}$ have

$$\star_{dx}^{-1}\partial_B f = \star_{dx}^{-1}(f'dx) = f', \quad f \in H_0^1(0,1),$$

and

$$(-\star_\omega^{-1}\partial_B, \mathcal{F}_B) = (\frac{d}{dx}, H_0^1(0,1)) \quad \text{"small"}$$

$$(\partial_B^\perp, \mathcal{D}_{\nu_\omega}(\partial_B^\perp)) = (\frac{d}{dx}, H^1(0,1)) \quad \text{"large"}$$

General situation:

Lemma (H./Schefer 2024)

We have $\mathcal{D}_{\nu_\omega}((\star_\omega^{-1}\partial_B)^*) = \mathcal{D}_{\nu_\omega}(\partial_B^\perp)$ *and*

$$(-\star_\omega^{-1}\partial_B)^* f = \partial_B^\perp f.$$

If in addition ω *is divergence free,* $\partial_B^*\omega = 0$*, then* $\mathcal{F} \subset \mathcal{D}_{\nu_\omega}(\partial_B^\perp)$*,*

$$\partial_B^\perp f = \star_\omega^{-1}\partial f, \quad f \in \mathcal{F},$$

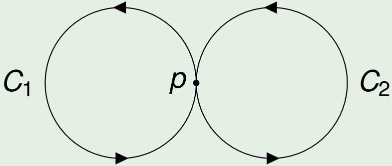
and $(\partial_B^\perp, \mathcal{F}_B)$ *is skew-symmetric on* $L^2(X, \nu_\omega)$*,*

$$\left\langle \partial_B^\perp f, g \right\rangle_{L^2(X,\nu_\omega)} = -\left\langle f, \partial_B^\perp g \right\rangle_{L^2(X,\nu_\omega)}, \quad f, g \in \mathcal{F}_B.$$

In general $\mathcal{F} \subsetneq \mathcal{D}_{\nu_\omega}(\partial_B^\perp)$, so that $\partial_B^\perp$ is beyond the "standard" analysis of $(\mathcal{E}, \mathcal{F})$.

Examples

Let X be the union of two circles C_i glued at a single point p and $\omega = \omega_1 dx_1 + \omega_2 dx_2 \in \ker \partial^*$ with $\omega_i \in \mathbb{R} \setminus \{0\}$, fix orientations, identify $H^1(C_i \setminus \{p\})$ with $H^1(0,1)$



$$\mathcal{D}(\partial^\perp) = \{f = (f_1, f_2) \in H^1(0,1) \oplus H^1(0,1) : \\ \omega_1(f_1(1) - f_1(0)) + \omega_2(f_2(1) - f_2(0)) = 0\}$$

(f_i discontinuous, but satisfy a weighted Kirchhoff condition)

Examples

$K = SG$, $h \in \mathcal{F}$ be harmonic on $K \setminus V_0$ with boundary values $h(q_0) = 1$ and $h(q_1) = h(q_2) = 0$ on V_0. Clearly $\partial h \in \ker \partial^*_{V_0}$. By Hino 2010 ∂h is minimal energy-dominant, can use $\star_{\partial h}$. Let $f_0, f_1, f_2 \in \mathcal{F}$ be such that

$$f_0(F_0q_1) = f_1(F_0q_1), \quad f_0(F_0q_2) = f_2(F_0q_2), \quad \text{but} \quad f_1(F_1q_2) \neq f_2(F_1q_2).$$

Then $f := \mathbf{1}_{K_0\setminus\{F_0q_1, F_0q_2\}} f_0 + \mathbf{1}_{K_1\setminus\{F_1q_2\}} f_1 + \mathbf{1}_{K_2} f_2$ is a bounded Borel function on K, discontinuous at F_1q_2, therefore not in $\mathcal{F}$. But $f \in \mathcal{D}(\partial^{\perp}_{V_0})$.

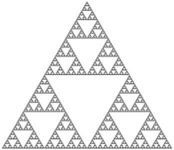
(cf. Li/Strichartz 2014, see also Schefer 2025).

- For time horizon $T=\infty$ and velocity field $b=\omega$ (CE) becomes

$$\begin{cases} \partial_t u(t)+\partial^{\perp}u(t)=0, & t>0, \\ u(0)=\mathring{u} \end{cases} \tag{CE'}$$

Assume that martingale dim of $(\mathcal{E},\mathcal{F})$ is one + that ω is minimal energy dominant and divergence free

- For $X=S^1$ have $\dim\ker\partial^*=1$ and (CE') is constant velocity case
- For SG $\ker\partial^*$ infinite dimensional, cf. Ionescu/Rogers/Teplyaev 2012, Cipriani/Guido/Isola/Sauvageot 2009, 2013 ... therefore (CE') interesting



Each loop around a hole ("lacuna") corresponds to a one-dim. subspace of $\mathcal{D}(\partial^*)$, base element is m.e.d. velocity field for circular flow

In the case of nonempty boundary $B = V_0$ also vector fields ∂h with suitable h harmonic on $SG \setminus V_0$ are divergence free (and m.e.d.)

Consider (CE') with fixed m.e.d. and divergence free velocity field ω.
Strategy:

1. Choose "adapted" volume measure $\nu = \nu_\omega$
2. "Choose" m-dissipative extension

$$-\star_\omega \partial \subset A \subset \partial^\perp$$

   on $L^2(X, \nu_\omega)$
   - Use characterization of m-dissipative extensions in terms of boundary quadruples by Arendt/Chalendar/Eymard 2023; see also Schubert/Seifert/Voigt/Waurick 2015, Waurick/Kaliske 2012
   - Our contribution: "topologically meaningful" boundary quadruples
3. Obtain well-posedness of (CE') in semigroup sense from Lumer-Phillips theorem; chosen extension encodes "boundary conditions"

A *boundary quadruple* for $-\star_\omega^{-1}\partial$ consists of pre-Hilbert spaces $H_\pm$ and linear maps $G_\pm:\mathcal{D}_{\nu_\omega}(\partial^\perp)\to H_\pm$ such that

$$\langle \partial^\perp f, g\rangle_{L^2(X,\nu_\omega)} + \langle f, -\partial^\perp g\rangle_{L^2(X,\nu_\omega)} = \langle G_+ f, G_+ g\rangle_{H_+} - \langle G_- f, G_- g\rangle_{H_-}$$

for all $f,g\in\mathcal{D}_{\nu_\omega}(\partial^\perp)$ (and some further conds).

Theorem (Arendt et al 2023, parametrization of all *m*-diss. extensions)

$-\star_\omega\partial\subset A\subset\partial^\perp$ is an m-dissipative extension $\Leftrightarrow$ there is a linear contraction $\Theta: H_-\to H_+$ such that

$$\mathcal{D}(A)=\{f\in\mathcal{D}_{\nu_\omega}(\partial^\perp):\ \Theta G_- f = G_+ f\}\quad \textit{and}\quad Af=\partial^\perp f,\ f\in\mathcal{D}(A).$$

Write A^Θ in this case.
Note: $\mathcal{D}(A^\Theta)=\mathcal{D}(A)$ obtained by restricting ”large” domain $\mathcal{D}_{\nu_\omega}(\partial^\perp)$.

Examples

Have $\mathcal{D}(\star_\omega^{-1}\partial) = H_0^1(0,1)$, $-\star_\omega^{-1}\partial f = -f'$, give $\mathcal{D}(\partial^\perp) = H^1(0,1)$ and $\partial^\perp f = -f'$.

Setting $H_+ := \mathbb{R}$, $H_- := \mathbb{R}$ and $G_+ f := f(0)$, $G_- f := f(1)$, obtain a boundary quadruple (H_-, H_+, G_-, G_+) for $-\star_\omega^{-1}\partial$.

$(A, \mathcal{D}(A))$ is an m-dissipative extension of $-\star_\omega^{-1}\partial$ if and only if

$$\mathcal{D}(A) = \{f \in H^1(0,1) :\ \theta f(1) = f(0)\} \quad \text{and} \quad Af = \partial^\perp f,\ f \in \mathcal{D}(A),$$

with some $-1 \leq \theta \leq 1$ and $Af = -f'$ for $f \in \mathcal{D}(A)$.

In sequel, additionally assume that

- $\mathbf{1} \in \mathcal{F}$ and $\ker \partial = \mathbb{R}$
- $\exists\, p \in X$ such that $(\mathcal{F}_{\{p\}}, \mathcal{E})$ is Hilbert and $\|f\|_{\sup} \leq c_p \mathcal{E}(f)^{1/2}$, $f \in \mathcal{F}_{\{p\}}$, here $c_p > 0$ universal constant

(... basically "$(\mathcal{E}, \mathcal{F})$ local regular resistance form, and X compact in resistance metric")

- Abstractly defined boundary quadruples always exist (Arendt et al 2023), but have no topo / geo meaning
- Aim: construct *meaningful* boundary quadruples (for metric graphs, SG, ...), so that boundary conditions can be understood
- First key tool: full description of the "large" domain $\mathcal{D}_{\nu_\omega}(\partial^\perp)$

Theorem (H./Schefer 2024, representation theorem)

For any $f \in \mathcal{D}_{\nu_\omega}(\partial^\perp)$ there are $g \in \mathcal{F}$, $u \in \mathcal{D}_{\nu_\omega}(\Delta)$ and $w \in \ker \partial^\perp$ such that

$$f = g + \star_\omega^{-1} \partial u + w,$$

and $z := \Delta u \in \ker \partial^\perp$.

Theorem says by which quantity f differes from an element g of $\mathcal{F}$

Examples

$X = [0,1]$ with $\omega = dx$ and $B = \{0,1\}$ find $f = g + u'$ with $g \in H^1(0,1)$ and $u' \in \mathbb{R}$. Here $\ker \partial^\perp = \{0\}$ and each harmonic function u is affine with constant slope $\star_{dx} du = u'$.

Examples

For the two circles glued at p find that
$z = (z_1, z_2), w = (w_1, w_2) \in \ker \partial^\perp \cong \mathbb{R}^2$.

- z_i describe jumps of components $f_i = f|_{C_i}$ of $f = (f_1, f_2)$ at p
- w_i describe average values of components f_i of f at p

Use representation theorem to obtain second key tool:

Theorem (H./Schefer '24, integration by parts)

For any $f_1, f_2 \in \mathcal{D}_{\nu_\omega}(\partial^\perp)$, with notation as above and $z_i := \Delta u_i$, have

$$\left\langle \partial^\perp f_1, f_2 \right\rangle_{L^2(X,\nu_\omega)} + \left\langle f_1, \partial^\perp f_2 \right\rangle_{L^2(X,\nu_\omega)} = \left\langle w_1, z_2 \right\rangle_{L^2(X,\nu_\omega)} + \left\langle z_1, w_2 \right\rangle_{L^2(X,\nu_\omega)},$$

indep. of choice of reps.

- "Boundary conditions" (loop conditions) involving w and z encode feedback of topological structure
- $\exists$ variant for nonempty finite B, it has further terms
- Proof by "straightforward manipulations"

Use ibp to construct boundary quadruples, sketch easy special case:

Suppose that $B = \emptyset$ and $\omega \in \ker \partial^*$. Consider $H_\pm := \ker \partial^\perp \ominus \mathbb{R}$, let $P : \ker \partial^\perp \to H_\pm$ denote orthogonal projection and set

$$G_\pm f := \frac{1}{\sqrt{2}} P(w \mp z).$$

Theorem (H./Schefer '24)

$(H_\pm, G_\pm)$ is a boundary quadruple for $\star_\omega^{-1}\partial$.

Using this result, obtain following:

Theorem (H./Schefer '24)

For any linear contraction $\Theta : H_\pm \to H_\pm$ and any $\mathring{u} \in \mathcal{D}(A^\Theta)$ the abstract Cauchy problem

$$\begin{cases} \partial_t u + \partial^\perp u = 0, & t > 0, \\ \Theta G_- u(t) = G_+ u(t) & t > 0, \\ u(0) = \mathring{u} \end{cases}$$

is well-posed.

Second line fixes enough information to have uniqueness. Weak solutions (cf. Ambrosio/Trevisan 2014) cannot be unique.
Now try to interpret Cauchy problem for simple enough ω and Θ (some special cases).

Examples

For single circle: $H_\pm = \{0\}$, $G_\pm = 0$, ibp is classical, (CE') has no extra condition, only periodic constant velocity solution, consistent with known uniqueness of operator extension.

Examples

For two circles glued at a single point p: $H_\pm \cong \mathbb{R}$,

$$G_\pm f = \frac{1}{2}\big(w_1 \mp z_1 - (w_\mp z_2)\big),$$

(CE') gets extra condition depending on parameter $\bar{\theta} \in [-1, 1]$, e.g.,

- $\bar{\theta} = 1$ periodic motion on one circle only
- $\bar{\theta} = -1$ periodic motion "changing tracks at p"

Examples (cont.)

$\mathcal{D}(A^\theta)$ is subspace of $H^1(0,1) \oplus H^1(0,1)$ consisting of all $f = (f_1, f_2)$ such that

$$\frac{1}{2}\begin{pmatrix} 1+\bar{\theta} & 1-\bar{\theta} \\ 1-\bar{\theta} & 1+\bar{\theta} \end{pmatrix}\begin{pmatrix} f_1(1) \\ f_2(1) \end{pmatrix} = \begin{pmatrix} f_1(0) \\ f_2(0) \end{pmatrix}$$

Many thanks !