

The impact of nonlinear projections on the dimension spectra
of fractal measures in Banach spaces

William Ott

University of Houston

International Seminar on Quantization, Geometry, and
Dynamics

Special Session 1

Outline

Background

Hilbert space case

Banach space case

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Banach space case

Primary question

\mathfrak{B} – Banach space

$A \subset \mathfrak{B}$ – compact set of interest

QUESTION – For a **typical** map $f : \mathfrak{B} \rightarrow \mathbb{R}^m$, do we have

$$\dim(f(A)) = \dim(A)?$$

MOTIVATION

- ▶ A – invariant set for an infinite-dimensional dynamical system
- ▶ Can we infer $\dim(A)$ from finite-dimensional data?

NOTE – f Lipschitz on $A \implies \dim(f(A)) \leq \dim(A)$

Finite-dimensional Banach spaces

$$\mathfrak{B} = \mathbb{R}^k$$

\dim_H – Hausdorff dimension

Theorem (Sauer/Yorke)

Let $A \subset \mathbb{R}^k$ be compact. For a *prevalent* \mathcal{C}^1 map $f : \mathbb{R}^k \rightarrow \mathbb{R}^m$,

$$\dim_H(f(A)) = \min \{m, \dim_H(A)\}.$$

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Thickness exponent

\mathfrak{H} – real Hilbert space

$A \subset \mathfrak{H}$ – compact set

Definition

The **thickness exponent** of A is the scaling exponent

$$\tau(A) = \limsup_{\varepsilon \rightarrow 0} \frac{\log(d(A, \varepsilon))}{\log(1/\varepsilon)},$$

where $d(A, \varepsilon)$ is the minimal dimension of finite-dimensional subspaces of \mathfrak{H} that ε -approximate A .

Thickness exponent: Intuition

$$\tau(A) = \limsup_{\varepsilon \rightarrow 0} \frac{\log(d(A, \varepsilon))}{\log(1/\varepsilon)}$$

- ▶ Any A : $\tau(A) \leq \dim_{\text{box}}(A)$
- ▶ Finite-dimensional disks
 - ▶ Thickness exponent zero
 - ▶ Arbitrarily high Hausdorff dimension
- ▶ $\{0, e_2/\log(2), e_3/\log(3), \dots\}$
 - ▶ Hausdorff dimension zero
 - ▶ ∞ = thickness exponent

Hilbert space result

\mathfrak{H} – real Hilbert space

$A \subset \mathfrak{H}$ – compact set

Theorem (Ott/Hunt/Kaloshin)

For a prevalent \mathcal{C}^1 map $f : \mathfrak{H} \rightarrow \mathbb{R}^m$,

$$\dim_H(f(A)) \geq \min \left\{ m, \frac{\dim_H(A)}{1 + \tau(A)/2} \right\}.$$

- ▶ This is sharp!
- ▶ Dimension preservation result when $\tau(A) = 0$

Non-computable attractors?

Ott/Hunt/Kaloshin estimate –

$$\dim_H(f(A)) \geq \min \left\{ m, \frac{\dim_H(A)}{1 + \tau(A)/2} \right\}$$

QUESTION Does there exist a **natural** attractor A with $\tau(A) > 0$ and $0 < \dim_H(A) < \infty$?

- ▶ 2D Naiver-Stokes attractor
 - ▶ Finite Hausdorff dimension
 - ▶ Thickness exponent zero (Friz/Robinson)

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A question of Robinson

QUESTION Does an Ott/Hunt/Kaloshin result hold for Banach spaces?

ISSUE

- ▶ \mathfrak{B} – Banach space
- ▶ F – finite-dimensional subspace of \mathfrak{B}
- ▶ How does the unit ball in F^* embed into \mathfrak{B}^* ?

Dual thickness exponent

\mathfrak{B} – real Banach space

$A \subset \mathfrak{B}$ – compact set

Definition

For $\theta > 0$, define the scaling exponent

$$\tau_{\theta}^*(A) = \limsup_{\varepsilon \rightarrow 0} \frac{\log(n_{\theta}(A, \varepsilon))}{\log(1/\varepsilon)},$$

where $n_{\theta}(A, \varepsilon)$ is the minimal dimension of linear subspaces

$V \subset \mathfrak{B}^*$ such that if $x, y \in A$ satisfy $|x - y| \geq \varepsilon$, then there exists

$\psi \in V$ with $\|\psi\| = 1$ and

$$|\psi(x - y)| \geq \varepsilon^{1+\theta}.$$

Dual thickness exponent $\longrightarrow \tau^*(A) = \lim_{\theta \rightarrow 0} \tau_{\theta}^*(A)$

First Banach space result

\mathfrak{B} – real Banach space

$A \subset \mathfrak{B}$ – compact set

Theorem (Ott/Stout/Zhou)

For a prevalent \mathcal{C}^1 map $f : \mathfrak{B} \rightarrow \mathbb{R}^m$,

$$\dim_H(f(A)) \geq \min \left\{ m, \frac{\dim_H(A)}{1 + \tau^*(A)} \right\}.$$

What about an estimate in terms of $\tau(A)$?

Thickness vs. dual thickness

Robinson –

$$\tau(A) = 0 \implies \tau^*(A) = 0$$

Ott/Stout/Zhou –

$$\tau^*(A) \leq \frac{\tau(A)}{1 - \beta\tau(A)}$$

β – projection exponent of the real Banach space

- ▶ $\beta \in [0, 1/2]$ (Kadec/Snobor)
- ▶ $\beta = 0$ in the real Hilbert space case
- ▶ $\beta = |1/q - 1/2|$ for L^q

Second Banach space result

\mathfrak{B} – real Banach space

$A \subset \mathfrak{B}$ – compact set

Theorem (Ott/Stout/Zhou)

For a prevalent C^1 map $f : \mathfrak{B} \rightarrow \mathbb{R}^m$,

$$\dim_H(f(A)) \geq \min \left\{ m, \frac{1 - \beta\tau(A)}{1 - \beta\tau(A) + \tau(A)} \dim_H(A) \right\}.$$

- ▶ Requires $\beta\tau(A) < 1$!
- ▶ Dimension preservation result when $\tau(A) = 0$