Sinai-Ruelle-Bowen Entropy: An Analogue of Boltzmann Entropy

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Feb. 15, 2026

Summary

Motivated by Gallavotti–Cohen Chaotic Hypothesis, we study whether the entropy of the Sinai-Ruelle-Bowen invariant measure of transitive Anosov systems can serve as an analogue of Boltzmann entropy in classic thermodynamics. In low dimensional systems, we prove properties that reflect the basic postulate and the second law of thermodynamics.

Entropy: A Brief History

Clausius (Rudolf Julius Emmanual): 1850s,

Entropy: can be expressed as a function of total internal energy U, Volume of gas V, and N: number of particles:

$$S = S(U, V, N).$$

Macroscopic description:
Entropy = En + Tropy ($\tau\rho o\pi\eta$: turning, transformation)

This entropy is only defined for systems at equilibrium.

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Boltzmann (Ludwig Edward): Statistical Interpretation of Entropy for both equilibrium and nonequilibrium systems: (1870s-1890s)

Velocities of particles are random variables:
They are described by a probability density function (3D) $f(\mathbf{v}, t)$.
This density function satisfies the Boltzmann equation:

$$\frac{\partial f}{\partial t} = Q(f, f).$$

Boltzmann Entropy:

$$H = \int \ln f(\mathbf{v}, t)\ f(\mathbf{v}, t) d\mathbf{v}$$

Boltzmann's H-Theorem:

$$S = -k_B H$$

Shannon (Claude Elwood): (1948)
A Mathematical Theory of Communication:
$\{p_1, p_2, \cdots, p_n\}$, a probability mass function: $p_i > 0, \sum p_i = 1$.

$$H = -K \sum_{i=1}^{n} p_i \log p_i,$$

derived based on a set of *three* Axioms.
H: measures the average information content (or uncertainty) of the source.
He used the name "entropy" because the formula resembles that of Boltzmann entropy.

Entropy: Justification

Shannon:
My greatest concern was what to call it. I thought of calling it 'information,' but the word was overly used. I decided to call it 'entropy,' because von Neumann told me, 'You should call it entropy for two reasons. In the first place your uncertainty function has been used in statistical mechanics under that name, so it already has a name. In the second place, and more important, nobody knows what entropy really is, so in a debate you will always have the advantage.

Entropy for a Dynamical System

Kolmogorov and Sinai: (1958)
(X, f) is a dynamical system:
Example: $f(x) = 2x \mod 1 : S^1 \to S^1$
μ : a probability measure on X, invariant under f:
$\mu(A) = \mu(f^{-1}(A))$. If f is invertible: $\mu(A) = \mu(f(A))$
$P_1 = \{B_i\}$: a partition of X.

$$H(\mu, P_1) = -\sum_i \mu(B_i) \ln \mu(B_i).$$

Refinements of the partition

$$P_n = \{B_{i_1} \cap f^{-1}(B_{i_2}) \cdots \cap f^{-n}(B_{i_n})\}$$

Entropy of f with respect to μ:

$$h_\mu(f) = \frac{1}{n} \lim_{n\to\infty} H(\mu, P_n).$$

KS Entropy played a huge role in modern theory of dynamical systems, especially, for chaotic systems:

Entropy > 0 implies Chaos!

But:
(1) Does it share anything with Boltzmann entropy, other than starting from a similar formula?

(2) Which μ, the invariant probability measure?

Simple Chaotic Dynamical Systems

1. Expanding maps on a circle $f : S^1 \to S^1$, $f'(x) > 1$.

2. Anosov systems on a 2-torus

$$\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{mod } 1$$

Sinai-Ruelle-Bowen Measure

m : Lebesgue measure.

$$\rho_f(A) = \lim_{n\to\infty} m(f^{-n}(A)).$$

Or,

$$\rho_f(A) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} m(f^{-k}(A)).$$

SRB Entropy: $h_{\rho_f}(f) := \mathcal{H}(f)$.

Chaotic Hypothesis

Galavotti-Cohen (1995) :

A reversible many particle system in a stationary state can be regarded as a transitive *Anosov* system for the purpose of computing the *macroscopic properties* of the system.

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Chaotic Hypothesis: extension

• *A thermodynamic system:* particles moving chaotically in an isolated chamber, evolving to its equilibrium.

$f(\mathbf{v}, t)$: Probability density for particle velocities.

• *A dynamical system model*: $\mathbf{x} \mapsto f(\mathbf{x}, t)$

$f(\mathbf{x}, t)$: possessing a unique SRB measure $\rho(\mathbf{x}, t)$, for each t.

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Macroscopic Properties of a Thermodynamic System

1. (Basic Postulate) An isolated system will attain, after sufficiently long time, an equilibrium state that is independent of its past history and characterized by its own intrinsic properties such as volume, energy, numbers of molecules of different type.

2. (Laws of Thermodynamics) 1st: Conservation of Energy; 2nd: Entropy cannot decrease;

Role of Boltzmann Entropy

1. $f(\mathbf{v}, t) \to f(\mathbf{v}, \infty)$: Maxwell - Boltzmann distribution (3D Gaussian). Boltzmann entropy is minimized at $f(\mathbf{v}, \infty)$.

2. $\frac{\partial H(f)}{\partial t} \leq 0$. Or, $\frac{\partial[-H(f)]}{\partial t} \geq 0$.

3. Boltzmann entropy *is* the sole 'reason' that the system evolves to its equilibrium.

4. $f(\mathbf{v}, t)$ is a 'generalized' gradient flow of the Boltzmann entropy.

Formula of the SRB Entropy

• For maps in the family of differentiable expanding maps on the circle:

$$f \in E^{C^2}(S^1) : f : \ S^1 \to S^1, f'(x) > 1.$$

Degree of f: n.
SRB entropy formula:

$$\mathcal{H}(f) = \int_{S^1} \ln f'(x)\rho_f(x)dx \leq \ln n,$$

the same as the Shannon entropy.
• For transitive Anosov maps on a compact manifold:

$$f \in A^{C^3}(M) : f : \ M \to M, Anosov.$$

SRB entropy formula:

$$\mathcal{H}(f) = \int_M \ln J^u f(x) d\rho_f \leq h_{\text{top}}(f),$$

the topological entropy of f.

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SRB Entropy Properties

Theorem: (J24)
1. $\mathcal{H}(f)$ is differentiable in f under a Sobolev norm. Its gradient flow exists: $\Phi(f_0, 0) = f_0$, $\Phi(f_0, t) = f_t$.

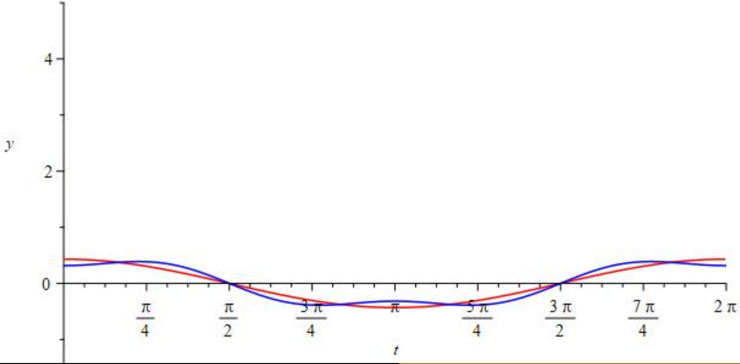
2. (Basic Postulate) $\lim_{t\to\infty} f_t = L$, the linear expanding map. $\mathcal{H}(f)$ reaches maximum $\ln n$ at L.

3. (2nd Law) $\frac{\partial \mathcal{H}(f_t)}{\partial t} > 0$.

The graph shows how a system approaches the equilibrium:
Red: Following the path given by the heat equation. frequency of the oscillation does not change:

$$e^{-t}\cos x \to 0.$$

Blue: Following the trajectory of the SRB entropy gradient flow: higher frequencies are created.



PDE connection

SRB entropy induced diffusion has close connection to this PDE:

$$u_t(y,t) = \frac{u_{yy}(y,t)}{u_y(y,t)}.$$

Ref. Gradient flow of the Sinai-Ruelle-Bowen entropy, *Comm. Math. Phys.* 405 (2024), (J)

Extension to Other Families of Uniformly Hyperbolic Systems

1. Family of Markov transformations on a closed interval (with Marco Lopez J. Stat. Phys. 2022) :
2nd Law.

2. Transitive Anosov maps on 2-torus:
2nd Law.

3. Families of expanding maps or transitive Anosov maps on n-torus: local existence of the SRB entropy gradient flow and gradient vector formula. (with Jiyan Chen, Contemporary Mathematics, 2026)