Rigidity of strong and weak foliations

Victoria Sadovskaya
The Pennsylvania State University

Joint work with Boris Kalinin

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Hyperbolic toral automorphisms and their perturbations

A matrix $L \in SL(d, \mathbb{Z})$ defines an automorphism L of the torus $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$

L is hyperbolic if it has **no** eigenvalues of modulus 1.

Stable and unstable subspaces:

$E^s =$ the sum of generalized eigenspaces of eigenvalues with modulus < 1
$E^u =$ . . . . > 1
$\mathbb{R}^d = E^s \oplus E^u$.

If a diffeomorphism $f : \mathbb{T}^d \to \mathbb{T}^d$ is C^1 close to L, then

- f is Anosov (or hyperbolic), i.e., there exists
  a continuous Df-invariant splitting $T\,\mathbb{T}^d = \mathcal{E}^s \oplus \mathcal{E}^u$ such that
  all vectors in $\mathcal{E}^s$ are contracted exponentially under the iterates of Df,
  all vectors in $\mathcal{E}^u$ . . . . Df^{-1}.
  $\mathcal{E}^s$ and $\mathcal{E}^u$ – stable and unstable sub-bundles.
- f is topologically conjugate to L, i.e., there is
  a homeomorphism h of $\mathbb{T}^d$ such that $L \circ h = h \circ f$.

$$\begin{array}{ccc} \mathbb{T}^d & \xrightarrow{\ f\ } & \mathbb{T}^d \\ h \downarrow & & \downarrow h \\ \mathbb{T}^d & \xrightarrow{\ L\ } & \mathbb{T}^d \end{array}$$

Conjugacy and foliations

L – a hyperbolic automorphism of $\mathbb{T}^d$, $\quad T\,\mathbb{T}^d = \mathbb{T}^d \times \mathbb{R}^d = E^s \oplus E^u$

f – a C^∞ diffeomorphism of $\mathbb{T}^d$ that is C^1 close to L, $\quad T\,\mathbb{T}^d = \mathcal{E}^s \oplus \mathcal{E}^u$

h – a topological conjugacy between f and L.

- h is not unique, but any two conjugacies have the same regularity.
- We consider the conjugacy close to the identity.
- h is bi-Hölder, but rarely smooth.

For L: E^s / E^u are tangent to linear stable/unstable foliations W^s / W^u.

For f: sub-bundles $\mathcal{E}^s$ / $\mathcal{E}^u$ are tangent to foliations $\mathcal{W}^s$ / $\mathcal{W}^u$.

- $\mathcal{W}^s$ and $\mathcal{W}^u$ are topological foliations with C^∞ leaves,
- h maps $\mathcal{W}^s$ to W^s, and $\mathcal{W}^u$ to W^u.

Dominated splitting

L – a hyperbolic automorphism of $\mathbb{T}^d$
with at least two distinct moduli of stable eigenvalues.

Example: $L \in SL(3,\mathbb{Z})$ has 3 eigenvalues, $0 < \lambda_{ss} < \lambda_{ws} < 1 < \lambda_u$.

Then L has a dominated splitting $E^s = E^{ss} \oplus E^{ws}$, and
$T\,\mathbb{T}^d = E^s \oplus E^u = E^{ss} \oplus E^{ws} \oplus E^u$.

f – a C^∞ diffeomorphism C^1 close to L. Then
$T\mathbb{T}^d = \mathcal{E}^s \oplus \mathcal{E}^u = \mathcal{E}^{ss} \oplus \mathcal{E}^{ws} \oplus \mathcal{E}^u$.

$\mathcal{E}^{ws}$ and $\mathcal{E}^{ss}$ – weak stable and strong stable subbundles.

- $\mathcal{E}^{ss}$ is tangent to $\mathcal{W}^{ss}$, which is a C^∞ subfoliattion of $\mathcal{W}^s$.
- $\mathcal{E}^{ws}$ is tangent to $\mathcal{W}^{ws}$,
  a subfoliation of $\mathcal{W}^s$ with only $C^{1+\text{Hölder}}$ leaves in general.

h – a topological conjugacy between f and L.

- $h(\mathcal{W}^{ws}) = W^{ws}$.
- Usually, $h(\mathcal{W}^{ss}) \neq W^{ss}$.

Goal: explore **rigidity** when $h(\mathcal{W}^{ss}) = W^{ss}$ or $\mathcal{W}^{ws}$ is smooth.

Rigidity of the strong foliation: $h(\mathcal{W}^{ss}) = W^{ss}$

Example: $L \in SL(3, \mathbb{Z})$ with eigenvalues $0 < \lambda_{ss} < \lambda_{ws} < 1 < \lambda_u$,
$f(x) = L(x) + \varphi(x) e_{ss}$ or $L(x) + \varphi(x) e_{ss} + \psi(x) e_u$.

Gogolev-Shi [2023]:

- $h(\mathcal{W}^{ss}) = W^{ss} \Leftrightarrow \mathcal{W}^u$ and $\mathcal{W}^{ss}$ are jointly integrable.
- For irreducible L under extra assumptions $\Leftrightarrow h$ is $C^{1+\text{Hölder}}$ along $\mathcal{W}^{ws}$.

Note: irreducible $\Rightarrow$ leaves of every L-invariant foliation are dense in $\mathbb{T}^d$.

Theorem (Kalinin-S)

L – hyperbolic automorphism of $\mathbb{T}^d$ with dense leaves of W^{ss}.
f – C^∞ diffeomorphism C^1 close to L. Then (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (2+) $\Rightarrow$ (3), and all are equivalent if L is (weakly) irreducible.

(1) $h(\mathcal{W}^{ss}) = W^{ss}$,

(2) $\mathcal{W}^u$ and $\mathcal{W}^{ss}$ are jointly integrable to a foliation $\mathcal{W}^{u+ss}$,

(2+) $\mathcal{W}^{u+ss}$ is C^∞ conjugate to the linear foliation $W^u \oplus W^{ss}$,

(3) $h^{ws} \in C^\infty(\mathbb{T}^d)$, and hence h is $C^{1+\text{Hölder}}$ along $\mathcal{W}^{ws}$.

h^{ws} is the E^{ws}-component of h w.r.to the splitting $E^u \oplus E^{ws} \oplus E^{ss}$.

Rigidity of the weak foliation: $\mathcal{W}^{ws}$ is smooth

- $h(\mathcal{W}^{ws}) = W^{ws}$. Since $h(\mathcal{W}^u) = W^u$,
  $\mathcal{W}^u$ and $\mathcal{W}^{ws}$ are jointly integrable to $\mathcal{W}^{u+ws} = h^{-1}(W^u \oplus W^{ws})$.
- Typically, the leaves of $\mathcal{W}^{ws}$ are only $C^{1+\text{Hölder}}$.
  Suppose $\mathcal{W}^{ws}$ is more regular.

Theorem (Kalinin-S)

L – hyperbolic automorphism of $\mathbb{T}^d$ with dense leaves of W^{ws}.
$r \notin \mathbb{N}$ is sufficiently large ($>$ ratio of top and bottom Lyapunov exponents of L on E^{ss}).
f – C^∞ diffeomorphism C^1 close to L. Then (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3).

(1) $\mathcal{W}^{ws}$ is a C^r subfoliation of $\mathcal{W}^s$,
(2) $\mathcal{W}^{u+ws}$ is C^r conjugate to the linear foliation $W^u \oplus W^{ws}$,
(3) $h^{ss} \in C^r(\mathbb{T}^d)$.

Rigidity of strong and weak foliations

From the previous two theorems:

- $h(\mathcal{W}^{ss}) = W^{ss} \iff$
  $\mathcal{W}^{u+ss}$ is C^∞ conjugate to $W^u \oplus W^{ws} \implies$
  $h^{ws} \in C^\infty(\mathbb{T}^d)$.
- $\mathcal{W}^{ws}$ is a C^∞ subfoliation of $\mathcal{W}^s \iff$
  $\mathcal{W}^{u+ws}$ is C^∞ conjugate to $W^u \oplus W^{ss} \iff$
  $h^{ss} \in C^\infty(\mathbb{T}^d)$.

Theorem (Kalinin-S)

L – hyperbolic automorphism of $\mathbb{T}^d$ with dense leaves of W^{ws} and W^{ss}.
f – C^∞ diffeomorphism C^1 close to L. Then (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3).

(1) $h(\mathcal{W}^{ss}) = W^{ss}$ and $\mathcal{W}^{ws}$ is C^∞ subfoliation of $\mathcal{W}^s$,
(2) $\mathcal{W}^u$ is conjugate to W^u by a C^∞ diffeomorphism of $\mathbb{T}^d$,
(3) $h^s \in C^\infty(\mathbb{T}^d)$.

C^∞ conjugacy h, symplectic L and f

Suppose for a hyperbolic automorphism L,

$T\,\mathbb{T}^d = E^s \oplus E^u = E^{ss} \oplus E^{ws} \oplus E^{wu} \oplus E^{su}$, and

the leaves of W^{ws}, W^{ss}, W^{wu}, W^{su} are dense in $\mathbb{T}^d$.

From the previous theorem:

If (i) $h(\mathcal{W}^{ss}) = W^{ss}$, $h(\mathcal{W}^{su}) = W^{su}$, and

(ii) $\mathcal{W}^{ws}$ and $\mathcal{W}^{wu}$ are C^∞ subfoliations of $\mathcal{W}^s$ and $\mathcal{W}^u$,

then h is a C^∞ diffeomorphism. The converse is clear.

Theorem (Kalinin-S)

Suppose L as above is symplectic with $\dim E^{ss} = \dim E^{su}$.

Let f be a C^∞ *symplectic diffeomorphism* C^1*-close to L.*

If $h(\mathcal{W}^{ss}) = W^{ss}$ *and* $h(\mathcal{W}^{su}) = W^{su}$, *then h is a* C^∞ *diffeomorphism.*

The assumption $\Rightarrow$ $\mathcal{E}^{u+ss}$ and $\mathcal{E}^{s+su}$ are C^∞

$\Rightarrow$ their intersection $\mathcal{E}^{ss+su}$ is C^∞.

Since $\mathcal{E}^{ss+su}$ and $\mathcal{E}^{ws+wu}$ are symplectic orthogonal,

$\Rightarrow$ $\mathcal{E}^{ws+wu}$ and hence $\mathcal{W}^{ws+wu}$ are C^∞

$\Rightarrow$ $\mathcal{W}^{ws} = \mathcal{W}^{ws+wu} \cap \mathcal{W}^s$ is a C^∞ subfoliation of $\mathcal{W}^s$, similarly for $\mathcal{W}^{wu}$.

Thank you!