Smooth rigidity for toral automorphisms

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Hyperbolic toral automorphisms and diffeomorphisms

A matrix $L \in SL(d, \mathbb{Z})$. The map $v \mapsto Lv$ on $\mathbb{R}^d$
projects to an automorphism L of the torus $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$.

L is hyperbolic or if L has **no** eigenvalues of modulus 1.

Stable and unstable subspaces:

$E^s =$ the sum of generalized eigenspaces of eigenvalues with modulus < 1
$E^u =$ . . . . > 1.
The vectors in E^s / E^u are contracted/expanded exponentially.
$\mathbb{R}^d = E^s \oplus E^u$.

A diffeomorphism f of $\mathbb{T}^d$ is Anosov (or hyperbolic) if there exist
a continuous Df-invariant splitting $T\,\mathbb{T}^d = \mathcal{E}^s \oplus \mathcal{E}^u$ such that

all vectors in $\mathcal{E}^s$ are contracted exponentially under the iterates of Df,

all vectors in $\mathcal{E}^u$ . . . . Df^{-1}.

$\mathcal{E}^s$ and $\mathcal{E}^u$ – stable and unstable sub-bundles.

Partially hyperbolic toral automorphisms and conjugacy

A matrix $L \in SL(d,\mathbb{Z})$. The map $v \mapsto Lv$ on $\mathbb{R}^d$ projects to an automorphism L of $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$.

L is partially hyperbolic if **some**, but not all, eigenvalues have modulus 1.
$\mathbb{R}^d = E^s \oplus E^c \oplus E^u$.

L is ergodic $\iff$ none of its eigenvalues is a root of unity.
Hyperbolic $\implies$ ergodic.

A diffeomorphism f of $\mathbb{T}^d$ is topologically conjugate to L if there is a homeomorphism H of $\mathbb{T}^d$ such that $L \circ H = H \circ f$.

$$\begin{array}{ccc} \mathbb{T}^d & \xrightarrow{f} & \mathbb{T}^d \\ {\scriptstyle H}\downarrow & & \downarrow{\scriptstyle H} \\ \mathbb{T}^d & \xrightarrow{L} & \mathbb{T}^d \end{array}$$

Let L be an ergodic automorphism of $\mathbb{T}^d$.
Then any two conjugacies between L and f have the same regularity.

Goal: bootstrap of regularity of a conjugacy H from C^1 or $C^{1+\text{Hölder}}$ to C^∞.

Conjugacy: basic facts and questions

Hyperbolic case.

- Any Anosov diffeomorphism of $\mathbb{T}^d$ is topologically conjugate to a hyperbolic automorphism L of $\mathbb{T}^d$.
- Any diffeomorphism f sufficiently C^1 close to a hyperbolic automorphism L is topologically conjugate to L.
- A conjugacy H is always bi-Hölder continuous, but usually not C^1, even if f is C^∞ and C^∞ close to L.

Question: Suppose a conjugacy between L and a C^∞ diffeo f is C^1. Is it C^∞?

Partially hyperbolic (PH) case.

- **PH** diffeomorphism is not necessarily conjugate to an automorphism.
- A small perturbation of **PH** L may not be topologically conjugate to it.

Question: Suppose there is a smooth conjugacy between L and a C^∞ diffeomorphism f. Is the conjugacy C^∞?

Hyperbolic case: prior results

- Two dimensions [de la Llave '92]
  Let f and g be C^∞ Anosov diffeomorphisms of $\mathbb{T}^2$
  topologically conjugate by H.
  If H and H^{-1} are absolutely continuous, then H is C^∞.
- Higher dimensional conformal [K.-Sadovskaya '03, de la Llave '04]
  Let $f, g : \mathcal{M} \to \mathcal{M}$ be C^∞ Anosov diffeomorphisms
  conformal on their stable and unstable subbundles, of dimension ≥ 2.
  Then any topological conjugacy between f and g is C^∞.
- General higher dimensional [de la Llave '92]
  For any $k \in \mathbb{N}$ there exists a reducible hyperbolic automorphism L
  and its C^∞ perturbation f such that
  the conjugacy between f and L is C^k but is not C^{k+1}.

Irreducibility of L is important.

L is irreducible if it has no nontrivial rational invariant subspaces,
equivalently, if its characteristic polynomial is irreducible over $\mathbb{Q}$.

Hyperbolic case: recent results

Theorem (K.-Sadovskaya-Wang, 2023, obtaining $C^{1+\text{Hölder}}$ regularity)

Let L be a hyperbolic automorphism of $\mathbb{T}^d$ (no irreducibility assumed).
let f be a $C^{1+Hölder}$ diffeomorphism.

Local: *f is C^1-close to L.*
If conjugacy between f and L is Lipschitz then it is $C^{1+Hölder}$.

Global: *No closeness assumption.*
If f is C^1 conjugate to L, then conjugacy is $C^{1+Hölder}$.

Theorem (K.-Sadovskaya-Wang, 2023, local hyperbolic result)

Let L be a (weakly) irreducible hyperbolic automorphism of $\mathbb{T}^d$,
let f be a C^∞ diffeomorphism $C^{r(L)}$ close to L.
If conjugacy between f and L is Lipschitz then it is a C^∞ diffeomorphism.

New results: hyperbolic and partially hyperbolic

Theorem (K.-Sadovskaya-Wang, hyperbolic global result)

Let L be a (weakly) irreducible hyperbolic automorphism of $\mathbb{T}^d$, let f be a C^∞ diffeomorphism of $\mathbb{T}^d$ (not necessarily close to L). If f and L are conjugate by a C^1 diffeomorphism H, then H is C^∞.

Theorem (K.-Sadovskaya-Wang, partially hyperbolic global result)

*Let L be a (weakly) irreducible **PH** ergodic automorphism of $\mathbb{T}^d$. let f be a C^∞ diffeomorphism of $\mathbb{T}^d$ (not necessarily close to L). If f and L are conjugate by a $C^{1+\text{Hölder}}$ diffeomorphism H, then H is C^∞.*

New approach. Works for hyperbolic and **PH** cases.

Ideas of the proof - I

L – (weakly) irreducible hyperbolic automorphism of $\mathbb{T}^d$.

f – C^∞ diffeomorphism conjugate to L via C^1 diffeomorphism H.

Then H is $C^{1+\text{Hölder}}$ by [KSW '23].

Can assume: H is homotopic to Id and $f(0) = H(0) = 0$.

Lift f and H to $\mathbb{R}^d$ as $Id + h$ and $L + r$.

$L \circ H = H \circ f \implies L \circ h = h \circ f + r \implies h = T(h) = L^{-1} \circ h \circ f + R$.

$\mathbb{R}^d = E^s \oplus E^u$ for L. For components in E^u,

$$h_u = T_u(h_u) = L_u^{-1}(h_u \circ f) + R_u.$$

T_u is a contraction on $C^0(\mathbb{T}^d, E^u) \implies h_u$ is its unique fixed point,

$$h_u = \sum_{k=0}^{\infty} L_u^{-k}(R_u \circ f^k). \qquad (\star)$$

Want: h_u is C^∞ (similarly h_s).

$\mathcal{W}^s$ and $\mathcal{W}^u$ – stable and unstable foliations for f tangent to $\mathcal{E}^s$ and $\mathcal{E}^u$.

$(\star)$ can be differentiated term-wise along $\mathcal{W}^s$ since Df^k decays.

So h_u is C^∞ along $\mathcal{W}^s$.

Ideas of the proof – II

Want: h_u is C^∞ along $\mathcal{W}^u$, **but** differentiating $\sum_{k=0}^{\infty} L_u^{-k}(R_u \circ f^k)$ along $\mathcal{W}^u$ yields a diverging series since Df^k grows.

Components along a Lyapunov subspace $E^\rho \subset E^u$: $h_\rho = \sum_{k=0}^{\infty} L_\rho^{-k}(R_\rho \circ f^k)$.
Differentiating along $\mathcal{W}^u$ still does not work.

Key idea: Instead analyze h_ρ as $-\sum_{k=0}^{\infty} L_\rho^k (R_\rho \circ f^{-k})$.
(!) The series does not converge in C^0.

- We differentiate once along the foliation $\mathcal{W}$ tangent to $\mathcal{E}_\rho \oplus \mathcal{E}_{faster}$
  obtain $D_{\mathcal{W}} h_\rho = -\sum_{k=1}^{\infty} L_\rho^k (D_{\mathcal{W}} R_\rho \circ f^{-k}) D_{\mathcal{W}} f^{-k}$ as a distribution.
- Higher derivative along $\mathcal{W}^u$: for any order m, $D_{\mathcal{W}^u}^m D_{\mathcal{W}} h_\rho \in L^2(\mathbb{T}^d)$.
- $\implies$ $D_{\mathcal{W}^u}^m h_\rho \in L^2(\mathbb{T}^d)$ for all m *(using irreducibility)*
- This $+$ h_ρ is C^∞ along $\mathcal{W}^s$ $\implies$ $h_\rho \in C^\infty(\mathbb{T}^d)$ [de la Llave].