Recurrence for pretentious systems along generalized Pythagorean triples (joint with N. Frantzikinakis)

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Partition regularity of linear equations

An old theorem of Schur [10] asserts the following:

Theorem (Schur, 1916)

The equation $x + y = z$ is partition regular, that is, if $\mathbb{N} = C_1 \cup \cdots \cup C_r$, then there are $x, y, z \in C_i$ such that $x + y = z$.

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A generalization of the previous is the following theorem of Rado [9]:

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The equation $ax + by = cz$ is partition regular (meaning that if $\mathbb{N} = C_1 \cup \cdots \cup C_r$, then there are $x, y, z \in C_i$ such that $ax + by = cz$) if and only if $a = c$ or $b = c$ or $a + b = c$ (in which case, (a, b, c) is called a Rado triple).

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Rado's theorem is in fact stronger, as it characterizes all systems of linear equations that are partition regular, but we did not include this more general form here.

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One might conjecture that (a, b, c) being a Rado triple is also sufficient for $ax^2 + by^2 = cz^2$ to be partition regular.

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In fact, they show density regularity for pairs, but we chose not state the theorem in this generality.

Ergodic-theoretic form of the problem

Definition

A multiplicative action is a quadruple $(X, \mathcal{X}, \mu, T_n)$*, where* $(X, \mathcal{X}, \mu)$ *is a probability space, and* $(T_n)_{n\in\mathbb{N}}$ *is a measure-preserving action of* $(\mathbb{N}, \times)$ *on* $(X, \mathcal{X}, \mu)$*.*

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Using Furstenberg's correspondence principle, the conjecture about partition regularity of $ax^2 + by^2 = cz^2$ follows from the following ergodic theoretic statement:

Conjecture

Let (a, b, c) *be a Rado triple. If* $(X, \mathcal{X}, \mu, T_n)$ *is a multiplicative action, and* $X = \bigcup_{j=1}^{r} T_j^{-1} A$*, then there are* $x, y, z \in \mathbb{N}$ *such that*

$$\mu(T_x^{-1} A \cap T_y^{-1} A \cap T_z^{-1} A) > 0 \quad \text{and} \quad ax^2 + by^2 = cz^2.$$

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Consider the space of completely multiplicative functions

$$\mathcal{M} = \{f : \mathbb{N} \to \mathbb{S}^1 : f(mn) = f(m)f(n) \quad \text{for all } m, n\},$$

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The space $\mathcal{M}$ can be decomposed as $\mathcal{M} = \mathcal{M}_{\text{pret}} \dot\cup \mathcal{M}_{\text{aper}}$, where

- $\mathcal{M}_{\text{pret}}$ consists of the functions that behave like "periodic" functions
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An action $(X, \mathcal{X}, \mu, T_n)$ is called pretentious if for every $F \in L^2(\mu)$, the spectral measure σ_F is supported of $\mathcal{M}_{\text{pret}}$, that is $\sigma_F(\mathcal{M}_{\text{aper}}) = 0$.

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Let (a, b, c) be a Rado triple and $(X, \mathcal{X}, \mu, T_n)$ be a pretentious multiplicative action. Then for any $A \subset X$ with $\mu(A) > 0$ and any $\varepsilon > 0$, there are infinitely many triples (x, y, z) such that

$$\mu(A \cap T_x^{-1}A \cap T_y^{-1}A \cap T_z^{-1}A) > (\mu(A))^4 - \varepsilon \quad \textit{and} \quad ax^2 + by^2 = cz^2.$$

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- For a general multiplicative action $(X, \mathcal{X}, \mu, T_n)$, the functions $F \in L^2(\mu)$ whose spectral measure is supported on $\mathcal{M}_{\text{pret}}$ form the pretentious factor of the system [1], so the previous theorem establishes multiple recurrence along generalized Pythagorean triples within this factor.

A consequence of our theorem

- Observe that for those structured systems, the statement is stronger than the conjectured one: we can add an extra copy of A in the recurrence statement, and also we only need a set of positive density, rather than a syndetic set A.

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As a consequence of our theorem, we get partition regularity for triples along partitions of $\mathbb{N}$ generated by pretentious multiplicative functions:

Corollary (Frantzikinakis-M., 2025)

If (a, b, c) is a Rado triple and $f_1, \ldots, f_r : \mathbb{N} \to \mathbb{S}^1$ are completely multiplicative and pretentious, then for each arc I in $\mathbb{S}^1$ containing 1, there are $x, y, z \in \mathbb{N}$ such that

$$f_j(x), f_j(y), f_j(z) \in I \text{ for } j = 1, \ldots, s \quad \text{and} \quad ax^2 + by^2 = cz^2.$$

Sketch of proof in case $a = b = c = 1$ - finitely generated

The solutions of $x^2 + y^2 = z^2$ are given in parametric form by

$x = P_1(m, n) = 2mn$, $y = P_2(m, n) = m^2 - n^2$, $z = P_3(m, n) = m^2 + n^2$.

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Hence, it suffices to prove that given $F \in L^\infty(\mu)$ with $F \geq 0$ and $\varepsilon > 0$, there are infinitely many $m, n \in \mathbb{N}$ such that

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We average over the grid $(Qm+1, Qn)$, where the averages over m, n are additive ($1 \leq m, n \leq N$), while over Q they are multiplicative ($Q \in \Phi_K$, where $(\Phi_K)_{K \in \mathbb{N}}$ is a multiplicative Følner sequence).

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$$\mathbb{E}_{m \leq N} |f(Qm+1) - 1| = \mathrm{o}_{K \to \infty, N \to \infty}(1) \tag{1}$$

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Using the (1),(2) above and the spectral theorem we get

$$\mathbb{E}_{m \leq N} \| T_{Qm+1} F - F \|_2 = o_{K\to\infty, N\to\infty}(1) \tag{3}$$

and $$\mathbb{E}_{m,n \leq N} \| T_{(Qm+1)^2 + (Qn)^2} F - F \|_2 = o_{K\to\infty, N\to\infty}(1). \tag{4}$$

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&\ge\Big(\int_X F\,d\mu\Big)^4+\mathrm{o}_{K\to\infty,N\to\infty}(1)
\end{aligned}$$

where the second equality follows from the Mean Ergodic Theorem for $(\mathbb{N},\times)$-actions (namely $\mathbb{E}_{Q\in\Phi_K}T_QF\to\int_X Fd\mu$ in $L^2(\mu)$), and in the last inequality we used Jensen's inequality.

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The reason is that in general a pretentious function does not necessarily concentrate at 1, so the concentration estimates are more complicated.

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The sets S_δ localize the the Archimedean characters, so for sufficiently small δ, the functions on the pretentious factor are sufficiently concentrated around themselves.

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- Then, we perform a triple multiplicative averaging over the Q_j, which eliminates the contribution from the oscillating components of F, to get a lower bound of the form

$$\int_X F \cdot F_1 \cdot F_2 \cdot F_3 \, d\mu - \varepsilon,$$

where each F_j is a projection of F on a suitable (non-oscillatory) factor of the original system.

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The fact that (a, b, c) is a Rado triple is necessary for the proof to work. This is expected, as the result is false without this assumption.

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The solutions are given in parametric form by

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In this case, even for an ergodic finitely generated action, if one runs our argument, they get a lower bound of the form $\int_X F d\mu \cdot \int_X F^2 \cdot T_2 F d\mu$, and the presence of $T_2 F$ breaks positivity.

Bibliography I

[1] N. Frantzikinakis. *Decomposition results for multiplicative actions and applications.* Preprint 2025, arXiv:2503.15175.

[2] N. Frantzikinakis and B. Host. *Higher order Fourier analysis of multiplicative functions and applications.* Journal of the American Mathematical Society, **30**(1): 67–157, 2017.

[3] N. Frantzikinakis, O. Klurman and J. Moreira. *Partition regularity of Pythagorean pairs.* Forum of Mathematics, Pi, **13**: Paper No. e5, 52, 2025.

[4] N. Frantzikinakis, O. Klurman and J. Moreira. *Partition regularity of generalized Pythagorean pairs.* Preprint 2024, arXiv:2407.08360.

[5] N. Frantzikinakis and A. Mountakis. *Recurrence for pretentious systems along generalized Pythagorean triples.* Preprint 2025, arXiv:2508.09778.

Bibliography II

[6] R. Graham. *Some of my favorite problems in Ramsey theory.* In Combinatorial number theory, pages 229–236. de Gruyter, Berlin, 2007.

[7] R. Graham. *Old and new problems and results in Ramsey theory.* In Horizons of combinatorics, volume **17** of Bolyai Soc. Math. Stud., pages 105–118. Springer, Berlin, 2008.

[8] O. Klurman, A. P. Mangerel, C. Pohoata and J. Teräväinen. *Multiplicative functions that are close to their mean.* Transactions of the American Mathematical Society **374**(11): 7967–7990, 2021.

[9] R. Rado. *Studien zur Kombinatorik.* Mathematische Zeitschrift, **36**(1):424–470, 1933.

[10] I. Schur. *Über die Kongruenz xm + ym zm (mod p).* Jahresbericht der Deutschen Mathematiker Vereinigung, **25**:114–117, 1916.

Thank you!