Smoothness of Markov Partitions for Expanding Toral Endomorphisms: Part 2

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Presentation Overview

1. Recap
2. Non-diagonalizable 2x2 matrices
3. Hausdorff Dimension

Disclaimer

We are novices in this area, and very much welcome feedback, comments, and references to related results!

Definitions

Let $A \in M_d(\mathbb{Z})$ such that for any eigenvalue λ of A we have $|\lambda| > 1$. A induces an *expanding toral endomorphism* $f_A : \mathbb{T}^d \to \mathbb{T}^d := \mathbb{R}^d / \mathbb{Z}^d$ defined by

$$f_A(x + \mathbb{Z}^d) = Ax + \mathbb{Z}^d.$$

We'll sometimes write f instead of f_A.

Definitions

Definition [Ashley-Kitchens-Stafford, 1992]

A *Markov partition* for f is a finite collection $\mathcal{M}$ of sets $R_i \subset \mathbb{T}^d$ such that for each $1 \leq i, j \leq |\mathcal{M}|$,

- $R_i = \overline{\text{int}\, R_i} \neq \emptyset$,
- $\text{int}\, R_i \cap \text{int}\, R_j \neq \emptyset \implies i = j$,
- $\mathbb{T}^d = \bigcup_{i=1}^n R_i$

and we have a *Markov property*:

- $f(\text{int}(R_i)) \cap \text{int}(R_j) \neq \emptyset \implies$ each $y \in \text{int}(R_j)$ has a unique preimage in R_i.

We denote the *boundary* of the partition $\partial\mathcal{M} := \bigcup_{i=1}^n \partial R_i$.

Properties

Note that

- Markov partitions exist, (see e.g. [Ruelle, 2004])
- $\partial\mathcal{M}$ is nowhere dense,
- $f(\partial\mathcal{M}) \subset \partial\mathcal{M}$.

We will say a Markov partition $\mathcal{M}$ is

- *linear* if $\partial\mathcal{M}$ is piecewise linear,
- *smooth* if $\partial\mathcal{M}$ is piecewise smooth,
- *hybrid* if $\partial\mathcal{M}$ is not smooth, but contains a smooth arc,
- *essentially nowhere smooth* if $\partial\mathcal{M}$ contains no C^1 arcs.

Symbolic Dynamics

Markov partitions are useful because they give a good *symbolic coding* of a system. Define a shift space

$$X = \{(x_0 x_1 \dots) \in \{1, \dots, k\}^{\mathbb{N}_0} : \forall i \in \mathbb{N}_0, f^{-1}\,\mathrm{int}(R_{x_i}) \cap \mathrm{int}(R_{x_{i+1}}) \neq \emptyset\},$$

with the shift map $\sigma : X \to X$ given by

$$\sigma(x)_i = x_{i+1}$$

Symbolic Dynamics

The expanding property of f gives a map $\pi : X \to \mathbb{T}^d$ defined as

$$\pi(x_0 x_1 \dots) := \bigcap_{i \in \mathbb{N}} f^{-i} R_{x_i}$$

which is well-defined, continuous, onto, makes the diagram

$$\begin{array}{ccc} X & \xrightarrow{\sigma} & X \\ \downarrow \pi & & \downarrow \pi \\ \mathbb{T}^d & \xrightarrow{f} & \mathbb{T}^d \end{array}$$

commute, and much more.

Main Theorem

Theorem.

Let A be a 2×2 expanding integer matrix. Then f_A admits a smooth Markov partition if and only if some power of A is diagonalizable with integer eigenvalues. In particular:

1. If A has irrational eigenvalues of different modulus, every Markov partition for f_A is essentially nowhere smooth.
2. If A has complex eigenvalues with irrational argument mod π, every Markov partition for f_A is essentially nowhere smooth.
3. If A is not diagonalizable, no Markov partition for f_A is smooth, but some Markov partitions may be hybrid.

Chayce covered the diagonalizable case. When A is not diagonalizable, the situation is qualitatively different.

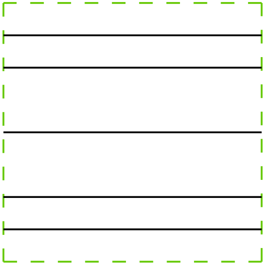
Jordan blocks

Lemma.

Let $k \in \mathbb{Z} \setminus \{-1, 0, 1\}$. The map $S : \mathbb{T}^1 \to \mathbb{T}^1$ with $Sx = k \cdot x \mod 1$ has a point with a dense orbit.

Jordan blocks

Let $A = \begin{pmatrix} k & 1 \\ 0 & k \end{pmatrix}$, and $\mathcal{M}$ a Markov partition for f_A.
Observe that $\partial\mathcal{M}$ must have a "vertical" part. Something like this



is not allowed: it violates the injectivty in the Markov property.

Jordan blocks

So, let γ be a C^1 curve contained in $\partial\mathcal{M}$, such that

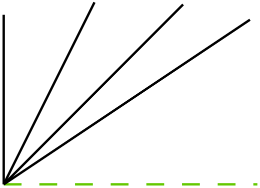
- γ' has a vertical component,
- the y-coordinate of $\gamma(0)$ has a dense orbit under $y \mapsto ky$ in $\mathbb{T}^1$.

Jordan blocks

For any $m \in \mathbb{N}$,

$$A^m = \begin{pmatrix} k & 1 \\ 0 & k \end{pmatrix}^m = \begin{pmatrix} k^m & mk^{m-1} \\ 0 & k^m \end{pmatrix}.$$

A acts on column vectors, so the ratio of the coordinates of $A^m\gamma'(0)$ tends to 0. That is, $\gamma'(0)$ gets stretched out horizontally:



Jordan blocks

Moreover, our choice for $\gamma(0) = (c_1, c_2)$ means that

$$A^m \gamma(0) = \begin{pmatrix} k^m c_1 + m k^{m-1} c_2 \\ k^m c_2 \end{pmatrix}$$

gets arbitrarily close in the y-direction to any point in $\mathbb{T}^2$.

Jordan blocks

Thus, we can approximate any point in the torus with $A^m\gamma(t)$ for some small t and large m, so

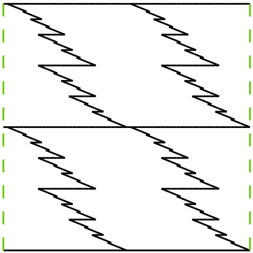
$$\gamma([0,1]) \subset \partial\mathcal{M}$$

has a dense orbit, a contradiction.

This means any curve in $\partial\mathcal{M}$ *with vertical change* cannot be C^1.

Hybrid behaviour

Using a construction from [Bedford, 1986], we can make a hybrid Markov partition for the matrix $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ which looks like this



Hybrid in higher dimensions

Consider the matrix

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Then any Markov partition $\mathcal{M}$ for $\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$ cannot be smooth, since its eigenvalues are $(5 \pm \sqrt{5})/2$. But if we take

$$\mathcal{M}' := \{R \times [0, 1/2] : R \in \mathcal{M}\} \cup \{R \times [1/2, 1] : R \in \mathcal{M}\},$$

we get a Markov partition for A that is linear in the z-coordinate.

Hausdorff Dimension

Theorem.

Let f_A be an expanding toral endomorphism, $\mathcal{M}$ a Markov partition for f_A, and λ the modulus of the smallest eigenvalue of A. Then

$$\dim_H(\partial\mathcal{M}) \leq \frac{h(f_A|_{\partial\mathcal{M}})}{\log\lambda}.$$

Here, $h(\cdot)$ denotes topological entropy.

Hausdorff Dimension

Proof idea:

- Make the partition small enough. f_A is locally bi-Lipschitz, so this won't change the Hausdorff dimension of $\partial\mathcal{M}$,
- Take the symbolic cover $\pi : X \to \mathbb{T}^d$,
- Generate a new SFT E on *pairs* of symbols corresponding to intersecting rectangles,
- This shift gives a cover $\pi' : E \to \partial\mathcal{M}$!

Hausdorff Dimension

A cylinder set in E corresponds to a pair of words $[w, w']$, and

$$\pi'([w, w']) = \pi([w]) \cap \pi([w']) \subset \pi([w]) \cup \pi([w']).$$

These cylinder sets form a cover of $\partial\mathcal{M}$, and by an argument of [Ashley-Kitchens-Stafford, 1992], we have

$$\operatorname{diam}(\pi([w])) \leq D\lambda^{-n}$$

whenever w is of length n large enough.

Hausdorff Dimension

Hence, if we denote $B_n(E)$ the number of words of length n in E, for fixed $s \in \mathbb{R}^+$, we have that

$$\begin{aligned}\sum_{(w,w')\in B_n(E)} \operatorname{diam}(\pi'([w,w']))^s &\leq \sum_{(w,w')\in B_n(E)} \operatorname{diam}(\pi([w]) \cup \pi([w']))^s \\ &\leq \sum_{(w,w')\in B_n(E)} \left(2D\frac{1}{\lambda^n}\right)^s \\ &\leq \#B_n(E)\left(2D\frac{1}{\lambda^n}\right)^s \\ &= (2D)^s \frac{C\mu^n + o(\mu^n)}{\lambda^{ns}}\end{aligned}$$

Where $\log(\mu) = h(E)$.

Hausdorff Dimension

If $s > \frac{\log \mu}{\log \lambda}$, then

$$\lim_{n \to \infty} (2D)^s \frac{C\mu^n + o(\mu^n)}{\lambda^{ns}} = 0.$$

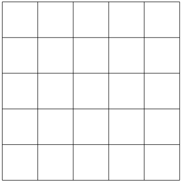
Moreover, π' is uniformly bounded-to-one, hence preserves entropy, i.e. $\log(\mu) = h(E) = h(f|_{\partial \mathcal{M}})$.

We conclude that

$$\dim_H(\partial \mathcal{M}) \leq \frac{\log \mu}{\log \lambda} = \frac{h(f|_{\partial \mathcal{M}})}{\log \lambda}.$$

Hausdorff Dimension

We can attain equality: Let $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ with the following Markov partition



This partition has $h(f|_{\partial\mathcal{M}}) = \log 2$.

Remarks

- The argument for bounding the Hausdorff dimension works for arbitrary expansive systems, not just toral endomorphisms,
- With some adjustments, it also works for invertible systems, e.g. hyperbolic toral automorphisms, where $\partial\mathcal{M}$ is generally not invariant

Thank you!

References

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