Smoothness of Markov Partitions for Expanding Toral Endomorphisms: Part 1

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Presentation Overview

1. Basic Setup + Background
2. Positive Result
3. Negative results

We're novices in this area; feedback is welcome!

Thanks to Mariusz Urbanski for pointing us to this problem and for his encouragement!

Definitions

Let $A \in M_d(\mathbb{Z})$ such that for any eigenvalue λ of A we have $|\lambda| > 1$. A induces an *expanding toral endomorphism* $f_A : \mathbb{T}^d \to \mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$ defined by

$$f_A(x + \mathbb{Z}^d) = Ax + \mathbb{Z}^d.$$

We'll sometimes write f instead of f_A.

Take $p : \mathbb{R}^d \to \mathbb{T}^d$ the projection map, so $fp = pA$.

Definitions

Definition (Ashley-Kitchens-Stafford, 1992)

A Markov partition for f is a finite collection $\mathcal{M}$ of sets $R_i \subset \mathbb{T}^d$ such that for each $1 \leq i, j \leq |\mathcal{M}|$,

- $R_i = \overline{\mathrm{int}(R_i)} \neq \varnothing$,
- $\mathrm{int}\, R_i \cap \mathrm{int}\, R_j \neq \varnothing \implies i = j$
- $\mathbb{T}^d = \bigcup_{i=1}^{k} R_i$

and we have a Markov property:

- $f(\mathrm{int}(R_i)) \cap \mathrm{int}(R_j) \neq \varnothing \implies$ *each* $y \in \mathrm{int}(R_j)$ *has a unique preimage in* R_i.

We denote the *boundary* of the partition $\partial\mathcal{M} := \bigcup_{i=1}^{k} \partial R_i$.

Properties

Note that

- Markov partitions exist, (see e.g. [Ruelle, 2004])
- $\partial\mathcal{M}$ is nowhere dense,
- $f(\partial\mathcal{M}) \subset \partial\mathcal{M}$.

We will say a Markov partition $\mathcal{M}$ is

- *linear* if $\partial\mathcal{M}$ is piecewise linear,
- *smooth* if $\partial\mathcal{M}$ is piecewise smooth,
- *hybrid* if $\partial\mathcal{M}$ is not smooth, but contains a smooth arc (this will show up in Huub's talk)
- *essentially nowhere smooth* if $\partial\mathcal{M}$ contains no C^1 arcs.

Symbolic Dynamics

Markov partitions are useful because they give a good *symbolic coding* of a system. Define a shift space

$$X = \{(x_0 x_1 \dots) \in \{1, \dots, k\}^{\mathbb{N}_0} : \forall i \in \mathbb{N}_0,\ f(\operatorname{int} R_{x_i}) \cap \operatorname{int} R_{x_{i+1}} \neq \varnothing\}$$

with the shift map $\sigma(x)_i = x_{i+1}$.

Symbolic Dynamics

The expanding property of f gives a map $\pi : X \to \mathbb{T}^d$ defined as

$$\pi(x_0 x_1 \ldots) := \bigcap_i f^{-i} R_{x_i}$$

which is well-defined, continuous, onto, makes the diagram

$$\begin{array}{ccc} X & \xrightarrow{\sigma} & X \\ \Big\downarrow{\pi} & & \Big\downarrow{\pi} \\ \mathbb{T}^d & \xrightarrow{f} & \mathbb{T}^d \end{array}$$

commute, and much more.

Background

- For *hyperbolic automorphisms* of $\mathbb{T}^3$, [Bowen, 1978] showed that Markov partitions are not smooth.
- [Cawley, 1991] extends this to a complete classification.
- Conjecture for expanding case is analagous:

Conjecture

$A \in M_d(\mathbb{Z})$ expanding admits smooth Markov partitions if and only if some power of A is diagonalizable over $\mathbb{Q}$.

Results

Theorem

Let A be a 2×2 expanding integer matrix. Then, f_A admits a smooth Markov partition if and only if some power of A is diagonalizable with integer eigenvalues. In particular:

1. *If A has irrational eigenvalues of different modulus, every Markov partition for f_A is essentially nowhere smooth.*
2. *If A has complex eigenvalues with irrational argument mod π, every Markov partition for f_A is essentially nowhere smooth.*
3. *If A is not diagonalizable, no Markov partition for f_A is smooth, but some Markov partitions may be hybrid.*

We'll cover 1 and 2 in this talk, then 3 in the next talk.

Positive Result

As we'll see, the positive direction of the previous theorem holds in all dimensions.

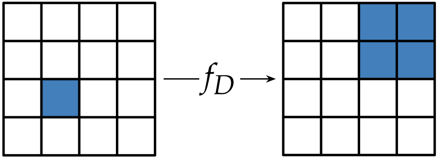
Proposition

Let $A \in M_d(\mathbb{Z})$ be an diagonalizable expanding matrix with integer eigenvalues. Then, A admits a linear (and hence smooth) Markov Partition $\mathcal{M}$.
Furthermore, $\mathcal{M}$ may be chosen such that f_A is injective on each $R \in \mathcal{M}$.

Sketch of Proof

Write $A = TDT^{-1}$, where $T, D \in M_d(\mathbb{Z})$ and D is diagonal. D has a very simple Markov partition: the cubes $\mathcal{C}$ in the $K^{-1}\mathbb{Z}^d$ lattice, when $K \geq |\det D|$.

Example: $D = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $K = 4$, and projecting $\mathcal{C}$ onto $\mathbb{T}^2$,



Note: for $C \in \mathcal{C}$, $DC = \bigcup_i^n C_i$ for some $C_i \in \mathcal{C}$.

Sketch of Proof

Fix $K > |\det D|$ to be a multiple of $\det T$. Our Markov partition is $pT\mathcal{C} := \{pTC' : C' \in \mathcal{C}\}$.

- f_A sends $pTC \in pT\mathcal{C}$ to unions of $pTC_i \in T\mathcal{C}$:

$$f_A(pTC) = pTDT^{-1}(TC) = pTDC = \bigcup_i^n pTC_i.$$

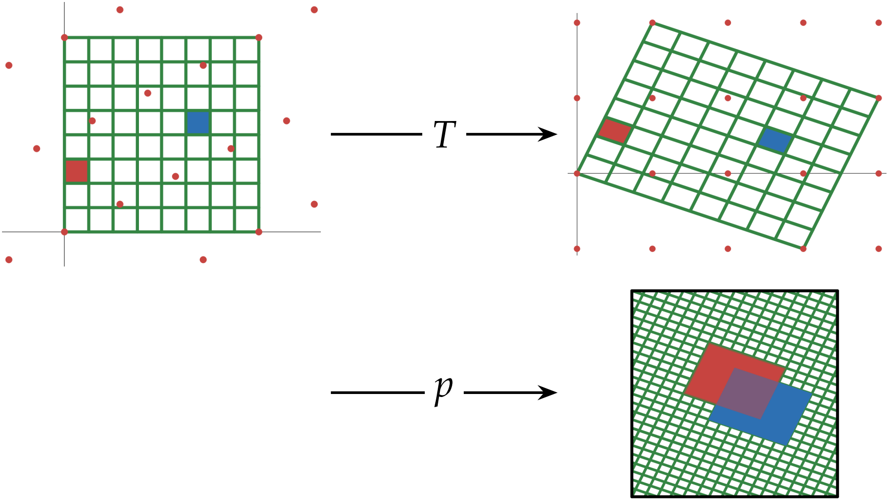
- Cubes are small, so f_A is injective on $R \in pT\mathcal{C}$.

$\implies$ $pT\mathcal{C}$ satisfies Markov condition!

Sticky point: making sure elements of $pT\mathcal{C}$ have disjoint interiors.

Sketch of Proof

Bad:



Sketch of Proof

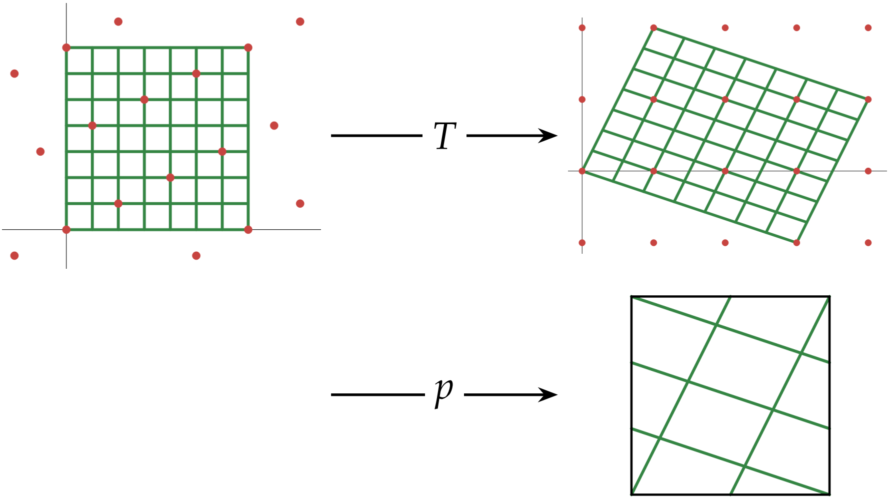
For $t \in \mathbb{Z}^d$, $T^{-1}(t) = \alpha/K$ for some $\alpha \in \mathbb{Z}^d$ since $\det T$ divides K. Hence, for $x \in \mathbb{R}^d$,

$$\underbrace{T(x) + t}_{\substack{\text{walking between} \\ p \text{ preimages}}} = \underbrace{T(x + \alpha/K)}_{\substack{\text{stepping across} \\ \text{`tiles' in } T\mathcal{C}}}$$

So for $x, y \in \mathbb{R}^d$, $p(x) = p(y)$ means that x, y lie in the same relative position of two 'tiles' in $T\mathcal{C}$.

Thus, $\operatorname{int}(pT\mathcal{C}) \cap \operatorname{int}(pT\mathcal{C}') \neq \varnothing$ iff $pT\mathcal{C} = pT\mathcal{C}'$. I.e. the elements of $pT\mathcal{C}$ have disjoint interiors! □

Sketch of Proof



Implications

Lemma (Markov Partition Refinement)

Let f be an expanding toral endomorphism. Then, for any $k \in \mathbb{N}$ and any Markov partition $\mathcal{M}$ of f^k such that f^k is injective on $R \in \mathcal{M}$, the set

$$\mathcal{E} = \left\{ E = R_0 \cap fR_1 \cap \ldots \cap f^{k-1}R_{k-1} : R_i \in \mathcal{M} \ \text{ and } \ \overline{\mathrm{int}(E)} = E \neq \varnothing \right\}$$

is a Markov partition for A.

Note: Refining a linear Markov partition like this produces another linear Markov partition.

Implications

So, combined with previous proposition, we get our positive result:

Theorem

If $A \in M_d(\mathbb{Z})$ is expanding and some power of A is diagonalizable with integer eigenvalues, then it admits a linear Markov Partition.

Negative Results, Real Eigenvalues

Proof.

- Let $A \in M_2(\mathbb{Z})$ be expanding with irrational eigenvalues $|\lambda_1| > |\lambda_2|$.
- Corresponding eigenvectors give irrational rays $\mathbb{R}^+ v_1$, $\mathbb{R}^+ v_2$
- Suppose $\partial\mathcal{M}$ contains a C^1 curve C
- Lift C to $\mathbb{R}^2$, parametrize it as $\gamma + x_0$ with $\gamma(0) = 0$

Fact: for all $\varepsilon > 0$, there exists an $M > 0$ such that $p([0, M]v_i + x)$ is ε-dense for any x

Sketch of Proof

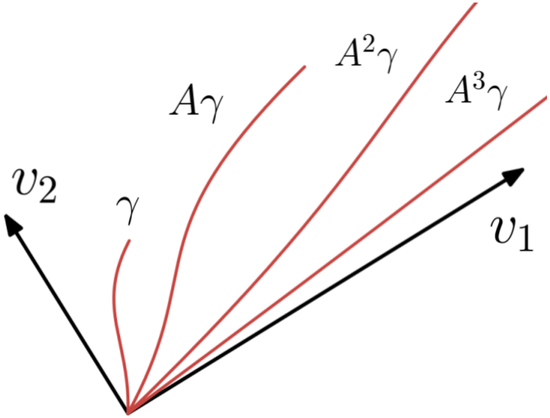
Hence, if a sub-arc of C is contained in a line parallel to v_1 or v_2 (i.e. γ' is co-linear with v_1 or v_2 on an interval), then $\partial\mathcal{M}$ is dense.

But $\partial\mathcal{M}$ is nowhere dense!

So, we can assume γ' is never co-linear with v_1 or v_2 (just restrict to a sub-arc where this is true).
Can take $\gamma'(t) \cdot v_1 > 0$. For simplicity, assume $\lambda_1 > 0$.

Sketch of Proof

Forward iterates $A^n\gamma$ look something like:



Sketch of Proof

So, A^n stretches γ towards the ray $v_1\mathbb{R}^+$. Since γ is C^1, this stretching is uniform in some sense.

So, given any $M > 0$, a sub-arc of $A^n\gamma$ can be made arbitrarily close to $[0, M]v_1$.

But for large M, $p([0, M]v_1 + x)$ is ε-dense, so $f^n C \subset \partial\mathcal{M}$ must be 2ε for large n! □

Sketch of Proof of Complex Eigenvalues case

For the complex case of our negative result, recall the following lemma:

Lemma

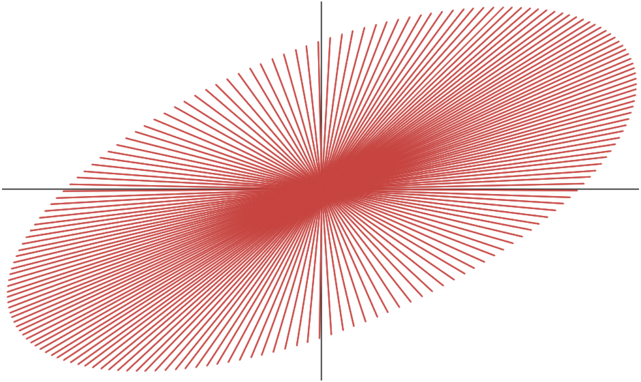
If $A \in M_2(\mathbb{R})$ has a complex eigenvalue $\lambda \notin \mathbb{R}$ with w the corresponding eigenvector (of the complexification of A), then A is given by

$$r \begin{bmatrix} u & v \end{bmatrix} R_{-\theta} \begin{bmatrix} u & v \end{bmatrix}^{-1}$$

where $r = |\lambda|$, $\theta = \arg(\lambda)$, $u = \mathit{Re}(w)$, $v = \mathit{Im}(w)$, and R_ϕ is the 2×2 rotation matrix.

Complex Eigenvalues Example

$A = \begin{bmatrix} 0 & 3 \\ -1 & 2 \end{bmatrix}$ has eigenvalues $1 \pm \sqrt{2}i$. $|1 + \sqrt{2}i| = \sqrt{3}$, and $\arg(1 + \sqrt{2}i)/\pi \notin \mathbb{Q}$. Plotting the lines to $\frac{1}{\sqrt{3^n}} A^n \begin{bmatrix} 1 \\ 1 \end{bmatrix}$:



Sketch of Proof

***Proof*.**

- Let $\theta = \arg(\lambda)$. For simplicity, suppose $A = rR_{-\theta}$, so $A^n = r^n R_{-n\theta}$.
- Suppose $\partial\mathcal{M}$ contains a C^1 curve, parametrize it as $x_0 + \gamma$
- Can choose $\gamma(0) = 0$, $\gamma(t) \neq 0$ for $t \neq 0$, and $\gamma' \neq 0$.
- Since $\gamma' \neq 0$, can continuously parametrize the standard angle ϕ of $\gamma(t)$:

$$\gamma(t) \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |\gamma(t)| \cos(\phi(t)).$$

Sketch of Proof

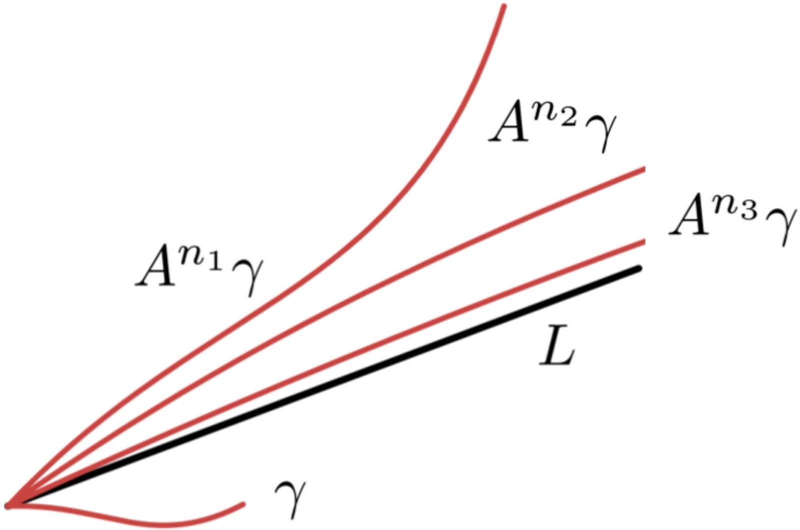
Let L be an irrational ray. Let α be the angle between L and the x-axis. With the density of $\{n\theta \mod 2\pi\}_n$, pick $n_k \to \infty$ such that

$$[\phi(0) - n_k\theta - \alpha] \mod 2\pi \searrow 0$$

$\gamma'(0)$ points along the line at angle $\phi(0)$ from the x-axis. So, $A^{n_k} = r^{n_k} R_{-n_k\theta}$ spins γ so that $A^{n_k}\gamma'(0)$ approaches the direction of L clockwise, and stretches γ along L.

Sketch of Proof

Pictorially, we have something like this:



Sketch of Proof

From here, proof concludes similarly to the real case:

If $u \in \mathbb{R}^2$ points along L, then recall that for all $\varepsilon > 0$, there exists an $M > 0$ such that $p([0, M]u + x)$ is ε-dense for any x

We 'stretch' along L, so $A^{n_k}\gamma([0,1])$ (and also $A^{n_k}(x_0 + \gamma([0,1]))$) is ε dense for k large. $\square$

That's All, Folks!

Stick around for more!

References

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