The extended Hausdorff dimension spectrum of a conformal iterated function system is maximal

Andrei Ghenciu, University of Wisconsin - Stout, Wisconsin, US
Ronnie Pavlov, University of Denver, Denver, Colorado, US

February 14, 2026

OVERVIEW

- Shift Spaces.
- β shifts.
- Conformal Iterated Function Systems (CIFSs).
- Limit set; Hausdorff dimension spectrum.
- Conformal Graph Directed Markov Systems.
- Shift generated iterated constructions.

OVERVIEW

- **QUESTION:**
  Given $0 \leq t \leq 1$, is there a subset A of the positive integers, so that the collection of all numbers that have continued fraction expansions with digits from A is a set of Hausdorff dimension t?
  Equivalent question: For every A a subset of the positive integers, let J_A be the collection of all the numbers in the unit interval whose continued fraction expansions have digits from A. Given $0 \leq t \leq 1$, is there a subset A_t of the positive integers so that the Hausdorff dimension of J_{A_t} is t?

OVERVIEW

- **More general question:**
  Consider a conformal iterated function system (CIFS) over an infinite alphabet I. For every A, a subset of I, let J_A be the limit set generated by A. Given a $0 \leq t \leq HD(J_I)$, is there a subset A_t of I, so that the Hausdorff dimension of J_{A_t} is t.

OVERVIEW

- **New question:**
  How can we properly phrase. or what is a good question when the alphabet is **finite**?

SHIFT SPACES

Let A be any finite or countable nonempty set, called the **alphabet**. The elements of A will be called **letters**.

Definition

A **word** over A is any finite sequence of letters from A. More formally, a word is any $\omega \in A^n$, for some $n \geq 1$; this n is referred to as the **length** of ω.

Definition

The **(one-sided) full A-shift** is the set of all infinite sequences of letters from A:

$$A^{\mathbb{N}} = \{x = (x_1, x_2, ..., x_i, ...), x_i \in A\}.$$

Thus, the full one-sided A-shift is just the collection of all infinite sequences with letters from A.

SHIFT SPACES

Definition

The **(left) shift map** σ from $A^{\mathbb{N}}$ is defined by $(\sigma x)_n = x_{n+1}$ for all $n \geq 0$, i.e.

$$(x_1, x_2, \ldots) \mapsto (x_2, x_3, \ldots).$$

Definition

A **shift space** or **subshift** over the alphabet A is a set $X \subset A^{\mathbb{N}}$ which is closed (in the product discrete topology) and invariant under σ, i.e. $x \in X \implies \sigma x \in X$.

SHIFT SPACES

Definition

A **nearest-neighbor shift of finite type** or **topological Markov chain** is a subshift over the alphabet A defined by a set $S \subset A^2$ of adjacencies in the following way:

$$X_S = \{(x_1, x_2, \ldots) \in A^{\mathbb{N}} \ : \ \forall n, (x_n, x_{n+1}) \in S\}.$$

A topological Markov chain over a finite alphabet is called **irreducible** if for all $i, j \in A$, there exists $w \in \mathcal{L}(X_S)$ beginning with i and ending with j. A topological Markov chain over a countable alphabet is called **finitely irreducible** if there exists N so that for all $i, j \in A$, there exists $w \in \mathcal{L}(X_S)$ with length at most N beginning with i and ending with j.

SHIFT SPACES

Definition

For a subshift X over A, the **language** of X is the set $\mathcal{L}(X)$ of all words occurring within points of x, i.e.

$$\mathcal{L}(X) = \{x_k \dots x_\ell \ : \ x \in X, k \le \ell \in \mathbb{N}\}.$$

For any n, the n-**language** of X is $\mathcal{L}_n(X) := \mathcal{L}(X) \cap A^n$, the set of n-letter words in $\mathcal{L}(X)$.

THE β-SHIFT

For $\beta > 1$, the β-shift X_β, is a subshift with alphabet $\{0, 1, \ldots, \lceil\beta\rceil - 1\}$.

It was originally defined by Parry as a symbolic coding of the linear map $x \mapsto \beta x \mod 1$.

The definition we give here uses greedy expansions with base β.

THE β-SHIFT

For any $t \in [0, 1)$, define the sequence $a(t)$ to be the greedy expansion of t in powers of β^{-1}: a_1 is the maximal integer for which $a_1\beta^{-1} < t$, and then for all $k > 1$, given $a_1, \ldots, a_k$, we define a_{k+1} as the maximal integer for which $a_{k+1}\beta^{-(k+1)} < t - \sum_{i=1}^{k} a_i\beta^{-i}$.

- **THE β-SHIFT X_β**

Then, the subshift X_β can be defined as the closure of the set of all such sequences $a(t)$. It is easily checked that a sequence x is in X_β iff for all $0 \leq i < j$, $\sum_{s=1}^{j-i} \frac{x_{s+i}}{\beta^s} < 1$.

ITERATED CONSTRUCTIONS

A **shift-generated iterated construction** is defined by a subshift X over an alphabet A with at least two letters, a non-empty compact metric space Y, a finite collection of non-empty compact subsets $\{Y_e\}_{e \in A}$ of Y, and a set of generators $\Phi = \{\phi_e : Y_e \to Y\}_{e \in A}$, where the ϕ_e's are one-to-one contractions which satisfy $\phi_f(Y_f) \subseteq Y_e$ whenever $ef \in \mathcal{L}_2(X)$.
Let $0 < s < 1$ be such that all these generators have a contraction ratio that does not exceed s. For every $\omega \in \mathcal{L}(X)$, set $Y_\omega = Y_{\omega_{|\omega|}}$ and

$$\phi_\omega : Y_\omega \to Y, \qquad \phi_\omega := \phi_{\omega_1} \circ \phi_{\omega_2} \circ \ldots \circ \phi_{\omega_{|\omega|}}.$$

Given $\omega \in X$, the compact sets $\phi_{\omega|_n}(Y_{\omega|_n})$, $n \geq 1$, are decreasing and their diameters converge to zero. More precisely,

$$\operatorname{diam}\big(\phi_{\omega|_n}(Y_{\omega|_n})\big) \leq s^n \operatorname{diam}(Y).$$

ITERATED CONSTRUCTIONS

This implies that the set

$$\bigcap_{n\geq 1} \phi_{\omega|_n}(Y_{\omega|_n})$$

is a singleton. We define the **coding map** $\pi : X \to Y$ by

$$\{\pi(\omega)\} = \bigcap_{n\geq 1} \phi_{\omega|_n}(Y_{\omega|_n})$$

and we define the limit set of the shift-generated construction to be

$$J(Y) = \pi(X).$$

ITERATED CONSTRUCTIONS

We call a shift-generated iterated construction **conformal** if the following conditions are satisfied:
i) There exists d so that for all $e \in A$, Y_e is a connected compact subset of $\mathbb{R}^d$ and $Y_e = \overline{\text{Int}_{\mathbb{R}^d}(Y_e)}$.
ii) (Open Set Condition (OSC)) For every $e, f \in A$, $e \neq f$,
$\phi_e(\text{Int}(Y_e)) \cap \phi_f(\text{Int}(Y_f)) = \emptyset$.
iii) For every $f \in A$, there exists a connected open set W_f with $Y_f \subset W_f \subset R^d$ so that the map ϕ_f extends to a C^1 conformal diffeomorphism of W_f into $\bigcap_{e \in A: ef \in \mathcal{L}_2(X)} W_e$.

(iv) There are two constants $L \geq 1$ and $\alpha > 0$ so that

$$\big||\phi_e'(x)| - |\phi_e'(y)|\big| \leq L\|(\phi_e')^{-1}\|^{-1} \cdot |x-y|^{\alpha}$$

for every $e \in A$ and for every pair of points $x, y \in Y_e$, where $|\phi_e'(x)|$ represents the norm of the derivative.

(v) (Cone Property) There exists $\gamma, l > 0$, $\gamma < \frac{\pi}{2}$ such that for every $x \in Y$ there exists an open cone $\mathrm{Con}(x, \gamma, l) \subseteq \mathrm{Int}(Y)$ with vertex x, central angle γ, and altitude l.

ITERATED CONSTRUCTIONS

As a consequence of (iv) we get the following:
(iv') (Bounded Distortion Property (BDP)) There exists $K \geq 1$ such that for all $\omega \in \mathcal{L}(X)$ and for all $x, y \in W_\omega$,

$$|\phi'_\omega(y)| \leq K|\phi'_\omega(x)|.$$

We note that in using our terminology, a **conformal iterated function system (CIFS)** is a shift-generated conformal iterated construction induced by a full shift (and $Y_e = Y$, for every $e \in A$), and a **conformal graph directed Markov system (CDGMS)** is a shift-generated conformal iterated construction induced by a topological Markov chain.

EXAMPLES

- **The Cantor set:** $A = \{0, 2\}$, $X = [0, 1]$, $\varphi_i(x) = \frac{1}{3}x + \frac{i}{3}$
- **Decimal expansions:** $A = \{0, 1, ..., 9\}$, $X = [0, 1]$, $\varphi_i(x) = \frac{1}{10}x + \frac{i}{10}$
- **Standard Continued Fractions:** $A = \mathbb{N}$, $X = [0, 1]$, $\varphi_i(x) = \frac{1}{i+x}$
- **Nearest Integer Continued Fractions:** $A = \mathbb{Z} \setminus \{-1, 0, 1\}$, $X_1 = [0, 1/2], X_2 = [-1/2, 0], X = [-1/2, 1/2]$, $\varphi_i(x) = \frac{1}{i+x}$

MAIN RESULTS

The following lemma is the main technical tool in the proofs of our main results.

Lemma

For any $\beta > 1$, $k \in \mathbb{N}$, $\delta < (1+\beta^{2k})^{1/2k} - \beta$, and $y \in X_{\beta+\delta}$, there exists a set $S \subset \mathbb{N}$ satisfying the following:
(1) S has gaps greater than k, i.e. $s \neq t \in S \implies |s-t| > k$
(2) $y(s) > 0$ for all $s \in S$
(3) The sequence x defined by replacing all letters of y at positions in S by 0s is in X_β.

MAIN RESULTS

Theorem

For any CIFS $\{\phi_i\}_{i\in A}$ *with* $A = \{0, \ldots, j\}$, *the Hausdorff dimension of* $J(X_\beta)$ *is continuous for* $\beta \in [0, j+1]$.

MAIN RESULTS

Theorem

For any CIFS $\{\phi_i\}_{i\in A}$ *on* Y *over a finite alphabet* A*, the extended Hausdorff dimension spectrum* $\{HD(J(X)) \;:\; X \subset A^{\mathbb{N}}\}$ *is* $[0, HD(J(Y))]$.

MAIN RESULTS

Theorem

For any CGDMS $\{\phi_i\}_{i\in A}$ *over a finite alphabet* A *defined by a topological Markov chain* $Z \subset A^{\mathbb{N}}$*, the restricted extended Hausdorff dimension spectrum* $\{HD(J(X)) \ : \ X \subset Z\}$ *is* $[0, HD(J(Z))]$.