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Research Article

Geometric Puzzles on a Circular Track: A Recreational Introduction to Quantization

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Abstract. Imagine a circular running track with several water stations placed around it. If only a few stations may remain open, or if a few new stations may be built, where should they be located so that runners travel the shortest possible distance to reach one? This simple question leads to a collection of recreational geometry puzzles involving distance, symmetry, optimization, Voronoi regions, and quantization. Quantization is the mathematical problem of replacing a large collection of points by a smaller set of representative points while minimizing the approximation error. Using only elementary mathematics, we investigate several configurations of points on a circle and discover elegant optimal arrangements. The article illustrates how sophisticated mathematical ideas can arise naturally from simple geometric games and puzzles, providing an accessible introduction to concepts that play important roles in geometry, optimization, data science, signal processing, and approximation theory.

Keywords. Recreational mathematics; geometry puzzles; quantization; circular distance; symmetry; Voronoi regions.

Mathematics Subject Classification (2020). 00A08; 51M04; 52C99; 94A29.

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1. Introduction

Many mathematical discoveries begin with simple questions. A puzzle, a game, or even an everyday situation can sometimes lead to surprisingly deep mathematics. Recreational mathematics has long served as a gateway to serious mathematical thinking by transforming ordinary observations into intriguing problems. The writings of Martin Gardner and Ross Honsberger have shown how playful problems can reveal elegant mathematical structures and lead naturally to deeper mathematical ideas [1, 2].

In this article we consider a simple circular-track puzzle. Imagine a circular running track with several water stations placed around it. Suppose that only a few stations may remain open. Which stations should be chosen so that runners do not have to travel too far?

A slightly different version is even more interesting. Suppose new stations may be built anywhere on the circular track. Where should they be placed?

At first glance, these questions appear practical rather than mathematical. However, they naturally lead to ideas involving distance, symmetry, optimization, and geometry.

These puzzles are elementary versions of problems from quantization theory, where one tries to approximate a collection of points by a smaller collection of representative points [3, 4]. Similar ideas appear in data compression, clustering, signal processing, and numerical approximation. The purpose of this article is not to develop advanced quantization theory. Instead, our goal is to present a friendly recreational introduction through geometric puzzles on a circle.

2. Distance on a Circle

Throughout this article, we consider a circle of circumference 1. Points on the circle will be represented by numbers in the interval $[0, 1)$. The points 0 and 1 represent the same point on the circle.

Definition 2.1. For two points $x, y \in [0, 1)$, the circular distance between x and y is

$$d(x, y) = \min\{|x - y|, 1 - |x - y|\}.$$

Thus $d(x, y)$ is the length of the shorter arc joining x and y.

Example 2.2. We have

$$d\left(0, \frac{1}{4}\right) = \frac{1}{4}.$$

Also,

$$d\left(0, \frac{3}{4}\right) = \frac{1}{4},$$

because the shorter way from 0 to $\frac{3}{4}$ goes backward around the circle.

3. The Average Squared Distance

Suppose the original stations are located at

$$x_1, x_2, \ldots, x_m.$$

Suppose we choose representative stations

$$a_1, a_2, \ldots, a_n.$$

Each original station is assigned to the nearest representative station. We measure the quality of the choice by the average squared distance

$$E(a_1, \ldots, a_n) = \frac{1}{m} \sum_{i=1}^{m} \min_{1 \le j \le n} d(x_i, a_j)^2.$$

The smaller the value of E, the better the representatives.

Definition 3.1. A set of n representative points is called optimal if it minimizes the average squared distance among all sets consisting of n representative points.

Remark 3.2. The square is used because points that are far away receive a larger penalty. This is common in approximation theory and quantization theory [3].

4. A First Puzzle: Two Opposite Stations

Suppose two stations are located at opposite points of the circle:

$$0 \text{ and } \frac{1}{2}.$$

Only one new representative station may be built. Where should it be placed? Figure 1 illustrates the two stations together with the optimal midpoint representatives.

Many readers first guess that one of the original stations should be chosen. Surprisingly, this is not the best choice.

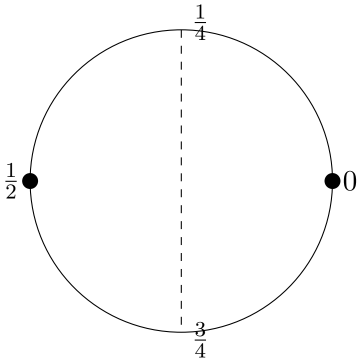

Figure 1: Two opposite stations and the two optimal midpoint representatives.

Proposition 4.1. *For the two opposite stations* 0 *and* $\frac{1}{2}$*, the optimal one-station representatives are* $\frac{1}{4}$ *and* $\frac{3}{4}$.

Proof. By symmetry, it is enough to consider a representative point x with

$$0 \le x \le \frac{1}{2}.$$

Then

$$d(x, 0) = x$$

and

$$d\left(x, \frac{1}{2}\right) = \frac{1}{2} - x.$$

The average squared distance is

$$E(x) = \frac{1}{2}\left[x^2 + \left(\frac{1}{2} - x\right)^2\right].$$

Expanding gives

$$E(x) = x^2 - \frac{1}{2}x + \frac{1}{8}.$$

Completing the square,

$$E(x) = \left(x - \frac{1}{4}\right)^2 + \frac{1}{16}.$$

Hence $E(x)$ is minimized when

$$x = \frac{1}{4}.$$

By symmetry, the point $\frac{3}{4}$ is also optimal. □

Remark 4.2. This example is surprising because the best representative is not one of the original stations. It lies halfway between them along one of the two semicircles.

5. The Triangle Puzzle

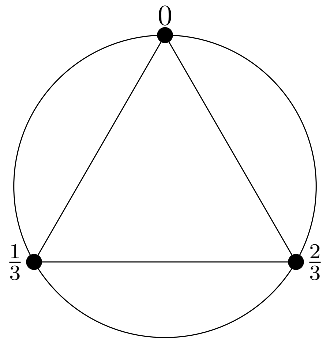

Figure 2: Three equally spaced stations forming an equilateral triangle on the circle.

Now consider three equally spaced stations:

$$0, \quad \frac{1}{3}, \quad \frac{2}{3}.$$

These points divide the circle into three equal arcs and form an equilateral triangle on the circle. The configuration is shown in Figure 2.

A natural question is:

> If only one representative station may be chosen, where should it be placed?

Because the three stations are perfectly symmetric, one might suspect that one of the original stations should be optimal. This is indeed the case.

Proposition 5.1. *For three equally spaced stations on a circle, each original station is an optimal one-station representative.*

Proof. By rotational symmetry, it is enough to compare representatives near the station at 0.

Let

$$0 \leq x \leq \frac{1}{6}.$$

The distances from x to the three stations $0, \frac{1}{3}$ and $\frac{2}{3}$ respectively are

$$x, \qquad \frac{1}{3} - x, \qquad \frac{1}{3} + x.$$

Hence the average squared distance is

$$E(x) = \frac{1}{3}\left[x^2 + \left(\frac{1}{3} - x\right)^2 + \left(\frac{1}{3} + x\right)^2\right].$$

Expanding and simplifying,

$$E(x) = \frac{1}{3}\left[3x^2 + \frac{2}{9}\right] = x^2 + \frac{2}{27}.$$

Since $x^2 \geq 0$, we obtain

$$E(x) \geq \frac{2}{27},$$

with equality if and only if

$$x = 0.$$

Thus the station at 0 is optimal.

By rotational symmetry, the stations

$$\frac{1}{3} \quad \text{and} \quad \frac{2}{3}$$

are also optimal. □

Remark 5.2. Unlike the case of two opposite stations, where the optimal representative lies exactly halfway between the stations, the symmetry of the equilateral triangle forces the optimal representative to coincide with one of the original stations.

6. The Square Puzzle

Consider four equally spaced stations:

$$0, \quad \frac{1}{4}, \quad \frac{1}{2}, \quad \frac{3}{4}.$$

These points divide the circle into four equal arcs and form a square on the circle. Figure 3 shows the square together with one of the midpoint representatives.

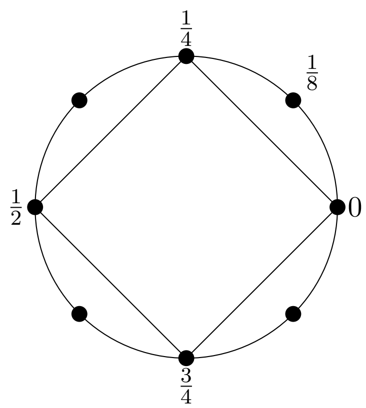

Figure 3: Four equally spaced stations (square) and one of the midpoint representatives.

Unlike the triangle puzzle, the square puzzle naturally leads to representative points that need not coincide with the original stations. Surprisingly, the following calculation shows that the average squared distance is minimized when the representative is placed halfway between two neighboring stations.

6.1. One Representative

Suppose that only one representative station may be built.

The four stations are equally spaced around the circle and therefore possess rotational symmetry. Every point on the circle is equivalent, under a suitable rotation, to a point in the interval

$$0 \le x \le \frac{1}{4}.$$

Consequently, it is sufficient to restrict attention to representatives in this interval. The distances from x to the four stations are

$$x, \qquad \frac{1}{4} - x, \qquad \frac{1}{2} - x, \qquad \frac{1}{4} + x.$$

Hence the average squared distance is

$$E(x) = \frac{1}{4}\left[x^2 + \left(\frac{1}{4} - x\right)^2 + \left(\frac{1}{2} - x\right)^2 + \left(\frac{1}{4} + x\right)^2\right].$$

Expanding and simplifying gives

$$E(x) = x^2 - \frac{1}{4}x + \frac{3}{32}.$$

Completing the square,

$$E(x) = \left(x - \frac{1}{8}\right)^2 + \frac{5}{64}.$$

Therefore the minimum within the interval

$$0 \le x \le \frac{1}{4}$$

occurs when

$$x = \frac{1}{8}.$$

By rotational symmetry, the corresponding points are

$$\frac{1}{8}, \qquad \frac{3}{8}, \qquad \frac{5}{8}, \qquad \frac{7}{8}.$$

Proposition 6.1.1. *For four equally spaced stations on a circle, the average squared distance is minimized when*

$$x = \frac{1}{8}.$$

Proof. The calculation above shows that

$$E(x) = \left(x - \frac{1}{8}\right)^2 + \frac{5}{64}.$$

Since the square term is nonnegative, the minimum value is attained precisely when

$$x = \frac{1}{8}.$$

Rotational symmetry then produces the corresponding arc-midpoint representatives. □

Remark 6.1.2. This result is rather surprising. In the triangle puzzle, the optimal representative coincides with one of the original stations. For the square puzzle, the optimal representative lies halfway between two neighboring stations.

7. Generalization for N Equally Spaced Stations

In this section, we generalize our previous results to an arbitrary number of $N \geq 2$ equally spaced stations on a circle of circumference 1. Let the original stations be located at $x_k = \frac{k}{N}$ for $k = 0, 1, \ldots, N-1$. By rotational and reflection symmetry, every point on the circle is equivalent to a representative point x lying in the fundamental domain:

$$0 \leq x \leq \frac{1}{2N}$$

We analyze the average squared distance function $E(x)$ by partitioning the N stations based on the shortest circular distance paths. Because the maximum possible circular distance on a unit circumference circle is $\frac{1}{2}$, the calculation of $d(x, x_k) = \min(|x - x_k|, 1 - |x - x_k|)$ splits naturally depending on whether the station index k lies on the forward or backward half of the track relative to x.

For a point $x \in [0, \frac{1}{2N}]$, the distance formulas evaluate as follows:

- **The Station at 0:** For $k = 0$, $d(x, 0) = x$.
- **Forward Stations:** For indices $1 \leq k \leq \left\lfloor \frac{N-1}{2} \right\rfloor$, the shorter arc to x travels forward, giving $d(x, x_k) = \frac{k}{N} - x$.
- **Backward Stations:** For indices where $x_k > \frac{1}{2}$, the shorter arc goes backward around the circle. Letting $k = N - j$, these correspond to $d(x, x_{N-j}) = \frac{j}{N} + x$ for $1 \leq j \leq \lfloor (N-1)/2 \rfloor$.
- **The Antipodal Station (Only if N is even):** For $k = N/2$, the station sits exactly opposite to 0 at $x_{N/2} = \frac{1}{2}$, yielding $d(x, \frac{1}{2}) = \frac{1}{2} - x$.

We now formally state and prove the optimization results as two separate theorems based on the parity of N.

7.1. The Case of an Odd Number of Stations

Theorem 7.1.1. *Let $N = 2m + 1$ be an odd natural number representing N equally spaced stations on a circle of circumference 1. The optimal one-station representatives are precisely the N original stations.*

Proof. Let $N = 2m + 1$ for an integer $m \geq 1$. The maximum index bounding both the forward and backward station groups is exactly $\lfloor N/2 \rfloor = \lfloor (2m + 1)/2 \rfloor = m$.

The average squared distance function $E(x)$ is defined as the sum of the squared distances to all stations divided by N:

$$E(x) = \frac{1}{2m+1} \left[x^2 + \sum_{k=1}^{m} \left(\frac{k}{N} - x \right)^2 + \sum_{j=1}^{m} \left(\frac{j}{N} + x \right)^2 \right]$$

Expanding the squared binomials inside both summations yields:

$$\left(\frac{k}{N} - x \right)^2 = \frac{k^2}{N^2} - \frac{2k}{N} x + x^2$$

$$\left(\frac{j}{N} + x \right)^2 = \frac{j^2}{N^2} + \frac{2j}{N} x + x^2$$

Substituting these back into the expression for $E(x)$, we group like terms:

$$E(x) = \frac{1}{N} \left[x^2 + \sum_{k=1}^{m} \left(\frac{k^2}{N^2} - \frac{2k}{N} x + x^2 \right) + \sum_{j=1}^{m} \left(\frac{j^2}{N^2} + \frac{2j}{N} x + x^2 \right) \right]$$

We now evaluate the sum of each component separately:

1. **The x^2 terms:** There is 1 initial x^2 term, m terms from the first sum, and m terms from the second sum. This gives a total of $1 + m + m = 2m + 1 = N$ terms:

$$\sum_{\text{all}} x^2 = Nx^2$$

2. **The linear x terms:** Combining the linear cross-terms from both summations over their identical indices yields:

$$\sum_{k=1}^{m} \left(-\frac{2k}{N}x\right) + \sum_{j=1}^{m} \left(\frac{2j}{N}x\right) = \sum_{k=1}^{m} \left(-\frac{2k}{N}x + \frac{2k}{N}x\right) = 0$$

   The linear x terms cancel out completely.

3. **The constant terms:** Combining the constant fractions from both sums gives:

$$\sum_{k=1}^{m} \frac{k^2}{N^2} + \sum_{j=1}^{m} \frac{j^2}{N^2} = 2\sum_{k=1}^{m} \frac{k^2}{N^2}$$

Bringing these components back into the main function:

$$E(x) = \frac{1}{N}\left[Nx^2 + 2\sum_{k=1}^{m} \frac{k^2}{N^2}\right] = x^2 + \frac{2}{N^3}\sum_{k=1}^{m} k^2$$

Because $\frac{2}{N^3}\sum_{k=1}^{m} k^2$ is a constant completely independent of x, minimizing $E(x)$ reduces purely to minimizing the remaining quadratic term, x^2. Since $x^2 \geq 0$ for all real numbers, the function achieves its unique global minimum inside the domain $[0, \frac{1}{2N}]$ at $x = 0$.

By rotational symmetry, every original station $x_k = \frac{k}{N}$ yields this exact minimum value $E(0)$. Thus, all N original stations are optimal one-station representatives when N is odd. □

7.2. The Case of an Even Number of Stations

Theorem 7.2.1. *Let $N = 2m$ be an even natural number representing N equally spaced stations on a circle of circumference 1. The optimal one-station representatives are precisely the N arc midpoints located at $\frac{2k+1}{2N}$ for $k = 0, 1, \ldots, N-1$.*

Proof. Let $N = 2m$ for an integer $m \geq 1$. The forward and backward station groups sum over the indices from 1 to $m-1$. Additionally, we must explicitly include the antipodal station sitting exactly opposite to 0 at $x_m = \frac{1}{2}$.

The average squared distance function $E(x)$ is given by:

$$E(x) = \frac{1}{2m}\left[x^2 + \left(\frac{1}{2} - x\right)^2 + \sum_{k=1}^{m-1} \left(\frac{k}{N} - x\right)^2 + \sum_{j=1}^{m-1} \left(\frac{j}{N} + x\right)^2\right]$$

Expanding the non-summed terms and the binomials inside the summations gives:

- $\left(\frac{1}{2} - x\right)^2 = \frac{1}{4} - x + x^2$
- $\left(\frac{k}{N} - x\right)^2 = \frac{k^2}{N^2} - \frac{2k}{N}x + x^2$
- $\left(\frac{j}{N} + x\right)^2 = \frac{j^2}{N^2} + \frac{2j}{N}x + x^2$

We compile the expanded components back into $E(x)$ to group like terms:

$$E(x) = \frac{1}{N}\left[x^2 + \left(\frac{1}{4} - x + x^2\right) + \sum_{k=1}^{m-1}\left(\frac{k^2}{N^2} - \frac{2k}{N}x + x^2\right) + \sum_{j=1}^{m-1}\left(\frac{j^2}{N^2} + \frac{2j}{N}x + x^2\right)\right]$$

We evaluate each grouped component separately:

1. **The x^2 terms:** There is 1 term from the station at 0, 1 term from the antipodal station, $(m-1)$ terms from the first sum, and $(m-1)$ terms from the second sum. This totals $1 + 1 + (m-1) + (m-1) = 2m = N$ terms:

$$\sum_{\text{all}} x^2 = Nx^2$$

2. **The linear x terms:** Just as in the odd case, the linear cross-terms within the two symmetric summations cancel each other out perfectly over the range 1 to $m-1$:

$$\sum_{k=1}^{m-1}\left(-\frac{2k}{N}x + \frac{2k}{N}x\right) = 0$$

However, the linear term $-x$ introduced by the antipodal station remains completely un-canceled. Thus, the total linear component is:

$$\sum_{\text{all}} \text{linear terms} = -x$$

3. **The constant terms:** Gathering the numerical constants yields:

$$\text{Constants} = \frac{1}{4} + 2\sum_{k=1}^{m-1}\frac{k^2}{N^2}$$

Combining these three evaluated blocks back into $E(x)$ gives:

$$E(x) = \frac{1}{N}\left[Nx^2 - x + \left(\frac{1}{4} + 2\sum_{k=1}^{m-1}\frac{k^2}{N^2}\right)\right]$$

$$E(x) = x^2 - \frac{1}{N}x + C_{\text{even}}$$

where $C_{\text{even}} = \frac{1}{4N} + \frac{2}{N^3}\sum_{k=1}^{m-1} k^2$ is a constant value independent of x. To isolate the point minimizing this quadratic function, we complete the square:

$$E(x) = \left(x - \frac{1}{2N}\right)^2 - \left(\frac{1}{2N}\right)^2 + C_{\text{even}}$$

$$E(x) = \left(x - \frac{1}{2N}\right)^2 + \left(C_{\text{even}} - \frac{1}{4N^2}\right)$$

Because the squared term $\left(x - \frac{1}{2N}\right)^2$ is strictly non-negative and reaches its minimum at zero, $E(x)$ is uniquely minimized within the fundamental domain when $x = \frac{1}{2N}$.

Recalling that $N = 2m$, this optimal coordinate simplifies precisely to the arc midpoint $x = \frac{1}{4m}$. Applying rotational symmetry across all N sectors of the track shifts this point to generate the complete set of optimal representatives located at the arc midpoints $\frac{2k+1}{2N}$ for $k = 0, 1, \ldots, N-1$. □

8. Voronoi Regions on a Circle

Once representative stations have been chosen, every point of the circle may be assigned to its nearest representative. This divides the circle into regions.

Definition 8.1. The Voronoi region of a representative point consists of all points on the circle that are at least as close to that representative as to any other representative.

Voronoi diagrams are important in geometry and optimization [6]. On a circle, they are especially easy to visualize.

For example, if the representatives are

$$0 \quad \text{and} \quad \frac{1}{2},$$

then each representative controls exactly one semicircle.

If the representatives are

$$0, \quad \frac{1}{3}, \quad \frac{2}{3},$$

then the circle is divided into three equal arcs.

Remark 8.2. Voronoi regions appear in city planning, wireless communication, biology, astronomy, computer graphics, and many other fields [6].

9. The Power of Symmetry

The examples above suggest a powerful idea.

> When a configuration has symmetry, optimal solutions often have related symmetry.

For example:

- for two opposite stations, the optimal one-station representative is an arc midpoint;
- for three equally spaced stations, the optimal one-station representative is one of the original stations;
- the same conclusion holds for five, seven, and, more generally, for any odd number of equally spaced stations;
- for four equally spaced stations, the optimal one-station representative is an arc midpoint;
- the same conclusion holds for six, eight, and, more generally, for any even number of equally spaced stations.

Symmetry is one of the central organizing principles of mathematics. It appears in geometry, algebra, number theory, physics, chemistry, computer science, and engineering.

In recreational mathematics, symmetry often turns a difficult-looking puzzle into a beautiful solution [1].

10. From Recreational Puzzles to Quantization Theory

The puzzles in this article are elementary. However, they point toward a much broader mathematical subject.

Suppose a large number of points must be replaced by a smaller number of representative points. How should the representatives be chosen so that the approximation is as accurate as possible?

This is the basic idea of quantization theory [3, 4].

Quantization appears in many areas, including:

- data compression,
- image processing,
- signal processing,
- clustering algorithms,
- numerical approximation,
- probability theory.

Geometric ideas related to packing, covering, and optimal arrangements also appear in the study of sphere packings and lattices [5].

Thus a simple puzzle about water stations on a circular track provides a small window into modern mathematical research.

11. Historical Note

Optimal quantization theory developed primarily during the twentieth century through its connections with information theory, signal compression, and probability [3, 4]. Today, quantization plays an important role in areas such as data compression, signal processing, machine learning, and numerical approximation.

Voronoi diagrams are named after the Russian mathematician Georgy Voronoi (1868–1908), whose work profoundly influenced the study of geometric partitions and optimal regions [6]. These diagrams now appear throughout mathematics and have applications in fields ranging from computer graphics and telecommunications to biology and astronomy.

At the same time, recreational mathematics has long provided an accessible pathway to deeper mathematical ideas. Through puzzles, games, and curious numerical or geometric phenomena, generations of students have encountered concepts that later developed into important areas of research. The writings of Martin Gardner and Ross Honsberger played particularly significant roles in popularizing recreational mathematics and demonstrating how simple questions can lead to rich mathematical discoveries [1, 2].

The present article follows this tradition by exploring elementary geometric puzzles on a circle that naturally introduce ideas related to symmetry, optimization, Voronoi regions, and quantization. Although the problems considered are recreational in nature, they provide a glimpse into mathematical concepts that continue to inspire contemporary research.

12. Challenges for the Reader

Exercise 12.1. For four equally spaced stations, compute the average squared distance when the representative is 0. Then compare it with the value obtained when the representative is $\frac{1}{8}$.

Exercise 12.2. For four equally spaced stations, show that the representatives $\frac{1}{8}$ and $\frac{5}{8}$ yield a minimum average squared distance of $\frac{1}{64}$.

Exercise 12.3. For a regular pentagon on the circle, investigate the best one-station representative.

Exercise 12.4. For a regular octagon on the circle, find a natural choice of two representative stations and compute the average squared distance.

Exercise 12.5. For six equally spaced stations on the circle, determine the optimal one-station representative directly, without using Theorem 7.2.1. Compare your answer with the even case of the general theorem.

Exercise 12.6. Create your own circular-track puzzle using seven stations that are not equally spaced.

13. Concluding Remarks

Recreational mathematics often reveals deep ideas hidden beneath simple questions. The puzzles in this article began with a circular running track and a few water stations. Yet this simple setting led naturally to distance, symmetry, optimization, Voronoi regions, and quantization.

One important lesson is that optimal solutions are not always the most obvious ones. For two opposite stations, the best representative is not an original station but a midpoint. For three equally spaced stations, the optimal representative coincides with one of the original stations. For four equally spaced stations, the best one-station representatives are arc midpoints. More generally, for any even number of equally spaced stations, the optimal one-station representatives are arc midpoints, whereas for any odd number of equally spaced stations, the optimal representative coincides with one of the original stations. These examples illustrate how symmetry can influence optimal configurations in different and sometimes surprising ways.

The ideas discussed here form a bridge between recreational mathematics and modern research. They show that even a simple geometric puzzle can lead to meaningful mathematical insight. Many natural extensions remain to be explored, including configurations with unequally spaced stations, multiple representative stations, alternative measures of approximation error, and analogous optimization problems on higher-dimensional spaces. We hope that these examples encourage readers to experiment with their own geometric puzzles, formulate conjectures, and discover how elementary questions can lead naturally to deeper mathematical ideas.

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- **Data availability:** Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.
- **Author Contributions:** All authors contributed equally in this manuscript.

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