ISQGD Recreational Mathematics Journal

A scholarly journal of recreational mathematics published by the International Society for Quality Growth and Development

Volume 1, Issue 1 (2026) • Article No. a003 • DOI: https://doi.org/10.67239/rmj.v01.n01.a003

Research Article

The Soaisu Structure of the 3×3 Magic Square

— Power Sum Coincidences Born from Symmetry —

Kenichi Takemura

Independent Researcher, Musashino, Tokyo, Japan

Corresponding author: mahora1123@gmail.com

ORCID: 0009-0004-3111-8826

Submitted: June 2, 2026 Revised: June 20, 2026

Accepted: July 4, 2026 Published: August 10, 2026

Abstract. This paper analyzes, from multiple perspectives, the **soaisu** structure hidden within the 3×3 magic square. We begin by clarifying the definition of soaisu and its relationship to the PTE problem, and reveal the soaisu symmetry formed by the rows and columns of the 3×3 magic square. We then apply domino tiling to both the magic square and the natural square, and investigate the soaisu formed by the resulting sum lists. Furthermore, we prove algebraically that the linear maps defined by the rows of the magic square generate infinitely many 3-3 soaisu ♡♡. Through these results, we demonstrate that the concept of soaisu serves as a powerful tool for analyzing the structure of magic squares.

Keywords. Soaisu; Prouhet–Tarry–Escott problem; magic squares; domino tiling; dihedral group symmetry.

Mathematics Subject Classification (2020). 11D72 (Primary); 05B45 (Secondary).

Citation. Kenichi Takemura, The Soaisu Structure of the 3×3 Magic Square — Power Sum Coincidences Born from Symmetry —, ISQGD Recreational Mathematics Journal, Volume 1, Issue 1 (2026), Article No. a003. DOI: https://doi.org/10.67239/rmj.v01.n01.a003

Main Results

Known facts and the position of this paper. The coincidence of squared sums for the outermost rows and columns of the 3×3 magic square is a property that has been individually observed in the context of multimagic square research (squares in which the power sums along every line all coincide). The concept of soaisu introduced in this paper extends the classical PTE setting — which focuses on two sets, consecutive exponents, and non-trivial solutions — to a unified framework accommodating any number of sets, arbitrary exponent ranges, and both trivial and non-trivial cases. For the general background on equal power-sum structures and the PTE problem, we refer the reader to the comprehensive survey by Chen Shuwen [1].

The new contributions of this paper are the following three points.

1. **The soaisu filter and D_4 symmetry (§3).** The fact that pairs of outermost rows (or columns) of the 3×3 magic square share their sum of squares can, in principle, be observed individually. In this paper, however, we reframe this within the unified framework of the "soaisu filter," and systematically show that the soaisu ♡♡ structure is invariant under all eight operations of the symmetry group D_4 of the square. To the best of the author's knowledge, describing the structure of magic squares through the combination of D_4 symmetry and the soaisu concept does not appear in prior literature.

2. **The connection between domino tilings and soaisu (§2).** The observation that domino tilings of the 3×3 natural square naturally produce a 4-4 soaisu ♡♡♡— an ideal solution of the PTE problem — applies and confirms in the order-3 case the method established for order-4 natural squares in the author's prior work [5]. This connection between the combinatorial operation of domino tiling and equal power-sum structures is a new perspective not found in the existing PTE literature.

3. **Infinite generation of 3-3 soaisu ♡♡ via linear maps (§4, Theorem 4.1.1).** We define linear maps f_i with coefficient vectors taken from each row of the 3×3 magic square, along with their reversed-order counterparts g_i $(i = 1, 2, 3)$, and prove algebraically that $\{f_1, f_2, f_3\}$ and $\{g_1, g_2, g_3\}$ always form 3-3 soaisu ♡♡, possibly trivial, for any real numbers (a, b, c). This shows that an infinite family of 3-3 soaisu ♡♡ can be systematically generated from a single magic square. We also note that the same holds when columns are used as coefficients, pointing to a row–column duality of the magic square that persists in the generation of soaisu.

Broader context. This paper restricts its discussion to the minimal 3×3 grid world, but it is becoming clear that soaisu structures are embedded with beautiful symmetry in larger natural squares and magic squares as well [3, 6]. Soaisu is anticipated to find further applications as a powerful and unified concept for describing and classifying these combinatorial structures.

1. Soaisu and the PTE Problem

1.1. Definition and Examples

Definition 1.1.1 (Soaisu)**.** Let $S_1, S_2, \ldots, S_m$ be m multisets[1] each of size n. If for all $l = 1, 2, \ldots, k$ we have

$$\sum_{a \in S_1} a^l = \sum_{a \in S_2} a^l = \cdots = \sum_{a \in S_m} a^l,$$

then we call this a **soaisu of strength** k of type n-n-$\cdots$-n (m copies)[3]. The strength k may also be expressed by the number of ♡ symbols. In this paper, strength 1 is written ♡, strength 2 as ♡♡, and strength 3 as ♡♡♡.

Note that two **identical** multisets, such as $(1, 1, 1)$ and $(1, 1, 1)$, also satisfy the definition. Since their k-th power sums agree for every k, we call such a pair a **trivial soaisu** (of strength ∞) and treat it separately. In this paper, trivial and non-trivial soaisu are distinguished explicitly, with the main emphasis placed on **non-trivial soaisu**, consisting of distinct multisets.

The term **soaisu** derives from the Japanese word "soai" (meaning "to cherish and love one another"), capturing the idea that multiple sets are bound together by the common tie of equal power sums; the syllable "su" means "number" in Japanese. In contrast to the historical name "Prouhet–Tarry–Escott problem," which refers to a precisely formulated problem in a specialist context, "soaisu" is intended to offer an intuitive and accessible term for a broader audience

[1] A multiset differs from an ordinary set in that the same element may appear more than once. For example, $\{5, 5, 15, 15\}$ is a multiset containing the element 5 twice and the element 15 twice, keeping track of these multiplicities.

who enjoy mathematics. In the context of recreational mathematics, where the joy of discovery is central, such a name can serve as a welcoming entry point for exploration.

1.2. The PTE Problem

Closely related to the definition of soaisu is the classical **Prouhet–Tarry–Escott problem** (hereafter, the PTE problem)[1].

PTE problem (Problem 1, [1]). Given a positive integer n, find two distinct sets of integers $\{a_1, a_2, \ldots, a_m\}$ and $\{b_1, b_2, \ldots, b_m\}$ such that

$$\sum_{i=1}^{m} a_i^k = \sum_{i=1}^{m} b_i^k \qquad (k = 1, 2, \ldots, n).$$

Here m is called the **size** and n the **degree** of the system.

It is known that no non-trivial solution exists when $m \leq n$, and when $m = n + 1$ (i.e., size exceeds degree by exactly one), a solution is called an **ideal solution of the PTE problem**[1, 2].

Where the PTE problem asks us to "find solutions to equal power sum equations of a given degree," soaisu is a concept that "gives a name to the very relationship of multiple sets sharing equal power sums." The PTE problem focuses on the special case of this relationship (two sets, consecutive exponents, non-trivial solutions), but soaisu can describe the situation uniformly regardless of the number of sets, the range of exponents, or whether the soaisu is trivial. In particular, as a tool for describing and discovering power sum symmetries lurking within combinatorial structures such as magic squares, the concept of soaisu is more expressive than the existing PTE vocabulary.

2. Soaisu in Action: The 3×3 Magic Square

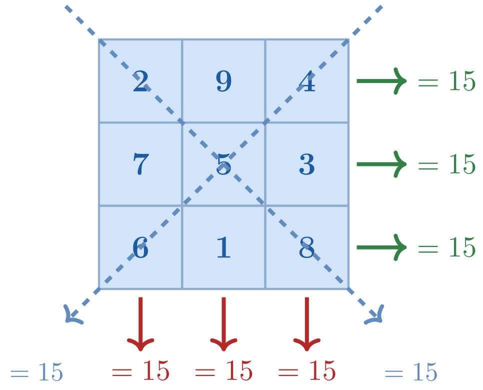
In the study of magic squares, beyond the fundamental property that the sums of every row, column, and diagonal coincide, there is a concept known as a **multimagic square**: a magic square in which the k-th power sums ($k = 1, 2, \ldots$) along every line all agree. An order-n magic square in which all line k-th power sums coincide is called a k**-multimagic square**.[2] Research on multimagic squares remains active today, but describing their conditions in full generality tends to require cumbersome notation. The concept of soaisu allows us to express these equal power-sum conditions intuitively and uniformly: the lines of a k-multimagic square of order n form an $n^{(2n+2)}$ soaisu of strength k, in a single concise phrase. In this section we take the 3×3 magic square as our stage and see how naturally soaisu structures arise.

2.1. The Protagonist: The 3×3 Magic Square

We begin by presenting the 3×3 magic square that serves as the stage for this paper. It is an arrangement of the integers 1 through 9 in which the sum of every row, column, and diagonal equals 15; up to rotation and reflection, there is essentially only one such square.

[2] For example, a 2-multimagic square (also called a bimagic square) is one in which the first and second power sums along every line both agree. No such square of order 3 exists; the smallest known order is 8.

3×3 magic square (row, column, and diagonal sums = 15)



2.2. The Eight Lines and the Soaisu They Form

The three numbers lying in each row, column, main diagonal, and anti-diagonal of the magic square together form eight 3-element sets.

Set	3 numbers	p_1 (1st power sum)	p_2 (2nd power sum)
Row 1	$\{2, 9, 4\}$	**15**	**101**
Row 2	$\{7, 5, 3\}$	**15**	83
Row 3	$\{6, 1, 8\}$	**15**	**101**
Column 1	$\{2, 7, 6\}$	**15**	**89**
Column 2	$\{9, 5, 1\}$	**15**	107
Column 3	$\{4, 3, 8\}$	**15**	**89**
Main diagonal	$\{2, 5, 8\}$	**15**	93
Anti-diagonal	$\{4, 5, 6\}$	**15**	77

The values of p_1 all agree at 15 (shown in bold). Therefore these eight sets form a

$$\text{3-3-3-3-3-3-3-3 soaisu } \heartsuit$$

(note that the p_2 values are not all equal, so the strength of the full eight-set soaisu does not exceed $\heartsuit$).

When the number of sets m is large, we abbreviate the $\underbrace{n\text{-}n\text{-}\cdots\text{-}n}_{m}$ soaisu as $n^{(m)}$ soaisu. For example, the eight sets of the 3×3 magic square form a $3^{(8)}$ soaisu $\heartsuit$.

In general, for an order-n magic square (one in which every row, column, and both diagonal sums are equal), the $2n + 2$ sets of n elements formed by the rows, columns, and both diagonals constitute an $n^{(2n+2)}$ soaisu $\heartsuit$.[3]

[3] The term "magic square" here refers to one satisfying the **standard definition**: all row, column, and diagonal sums are equal. Under this condition, the first-power sums of all $2n + 2$ lines coincide, and thus $n^{(2n+2)}$ soaisu $\heartsuit$ holds.

It is important to note that for $n \geq 4$ there exist **semi-magic squares** in which the row and column sums are equal but the diagonal sums differ. For instance, in a 4×4 semi-magic square the row and column sums all equal 34, but the two diagonal sums may differ from 34 (e.g., 30 and 38). In such cases, $\heartsuit$ does not hold for the full collection of $2n + 2$ lines including the two diagonals.

Furthermore, the 3×3 magic square that is the protagonist of this paper is essentially unique up to rotation and reflection, and all of its row, column, and diagonal sums equal 15. The claim of this section therefore applies fully in this case.

2.3. Domino Tiling of the Magic Square: An Example of Trivial Soaisu

We cover 8 of the 9 cells with 4 dominoes (1×2 tiles), leaving one cell uncovered. Excluding the central cell (value 5), there are exactly **two** distinct tilings.

2	9	4
7	5	3
6	1	8

M_1
(2+9)(4+3)(7+6)(1+8)
$W = \{11, 7, 13, 9\}$

2	9	4
7	5	3
6	1	8

M_2
(2+7)(9+4)(3+8)(6+1)
$W = \{9, 13, 11, 7\}$

Comparing the two tilings, the sum lists of the paired numbers within each domino are both $W = \{7, 9, 11, 13\}$, in perfect agreement. In other words, two tilings that look entirely different produce exactly **the same set** as their output.

Since identical sets have equal k-th power sums for every k, this is an example of a 4-4 soaisu $\heartsuit^{(\infty)}$, i.e., a **trivial soaisu**.

At first glance, the fact that "both yield the same set" might seem unexciting, but this hints at the deep symmetry that the 3×3 magic square contains within itself.

2.4. Domino Tiling of the 3×3 Natural Square

An order-n natural square is an $n \times n$ array in which the consecutive natural numbers 1 through n^2 are placed in row-major order (left to right, top to bottom). The 3×3 natural square is as follows.

1	2	3
4	5	6
7	8	9

3×3 **natural square**

Let us attempt the same domino partition of the outer cells that we performed in the previous section, and see what happens.

1	2	3
4	5	6
7	8	9

U_1
(1+2)(3+6)(4+7)(8+9)
$W = \{3, 9, 11, 17\}$

1	2	3
4	5	6
7	8	9

U_2
(1+4)(2+3)(6+9)(7+8)
$W = \{5, 5, 15, 15\}$

Tiling	Sum list W	p_1	p_2	p_3	p_4
U_1	$\{3, 9, 11, 17\}$	**40**	**500**	**7000**	104804
U_2	$\{5, 5, 15, 15\}$	**40**	**500**	**7000**	102500

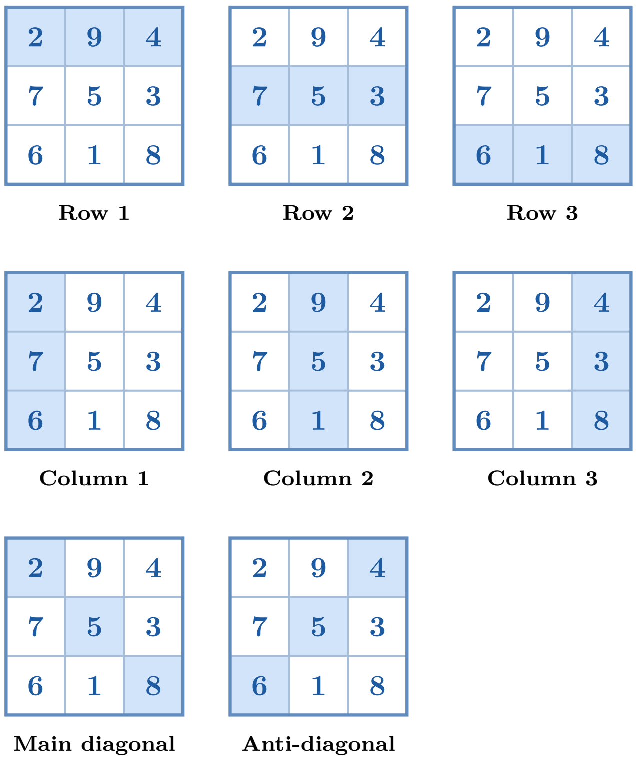
Remarkably, p_1, p_2, p_3 (in bold) all agree (note that p_4 does not: $104804 \neq 102500$). Therefore $\{3, 9, 11, 17\}$ and $\{5, 5, 15, 15\}$ form a **4-4 soaisu** ♡♡♡ (an ideal solution of the PTE problem).

This is a surprising fact: a natural square formed by simply arranging 1 through 9 in order naturally produces, through the combinatorial operation of domino tiling, an ideal solution to the notoriously difficult PTE problem of number theory.

3. The Soaisu Structure of the 3×3 Magic Square

3.1. Power Sums of the Eight Lines

We tabulate the values p_1, p_2, p_3 for the 3-element sets corresponding to each line (rows, columns, and diagonals) of the 3×3 magic square.



Line	3 numbers	p_1	p_2	p_3	Strength
Row 1	$\{2, 9, 4\}$	**15**	**101**	801	3-3 soaisu ♡♡
Row 3	$\{6, 1, 8\}$	**15**	**101**	729	
Row 2	$\{7, 5, 3\}$	**15**	83	495	
Column 1	$\{2, 7, 6\}$	**15**	**89**	567	3-3 soaisu ♡♡
Column 3	$\{4, 3, 8\}$	**15**	**89**	603	
Column 2	$\{9, 5, 1\}$	**15**	107	855	
Main diagonal	$\{2, 5, 8\}$	**15**	93	645	
Anti-diagonal	$\{4, 5, 6\}$	**15**	77	405	

The value p_1 equals 15 for every line (the 8 sets form a $3^{(8)}$ soaisu ♡). As for p_2, Row 1 and Row 3 (both 101) agree, and Column 1 and Column 3 (both 89) agree, each pair forming a **3-3 soaisu ♡♡**. On the other hand, p_3 agrees for none of the pairs, so the maximum strength achieved is ♡♡. We also note here that for the two diagonals, p_2 satisfies $93 \neq 77$, and moreover this value does not coincide with p_2 for any other line.

3.2. The Soaisu Filter and Its Rotational Symmetry

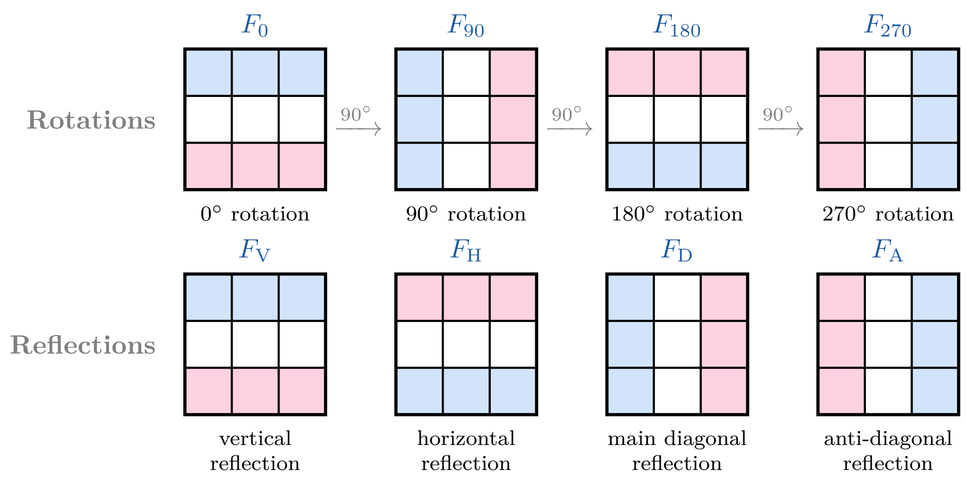
To express the ♡♡ structure inherent in the 3×3 magic square concisely, we introduce the concept of a **soaisu filter**.

A filter is a "mask" in which each cell of a 3×3 grid is colored one of three colors: blue, pink, or white. When the filter is overlaid on the magic square, it encodes the correspondence that the **three numbers in blue cells** and the **three numbers in pink cells** form a 3-3 soaisu ♡♡.

As a starting point, we define filter F_0 (the 0° rotation, i.e., the original orientation) by coloring Row 1 blue, Row 3 pink, and Row 2 white. As verified in the previous section, applying this filter to the magic square gives $\{2, 9, 4\}$ (blue) and $\{6, 1, 8\}$ (pink) as a ♡♡ pair.

We now describe the **symmetry transformations of the square**. There are exactly eight ways to superimpose a 3×3 square grid onto itself without distortion: 4 **rotations** (0°, 90°, 180°, 270°) and 4 **reflections** (horizontal, vertical, main diagonal, anti-diagonal). In mathematics, these are collectively called the dihedral group D_4.

The results of applying each of these eight transformations to filter F_0 are shown below (top row: 4 rotations; bottom row: 4 reflections).



The following table confirms that for all eight filters, the three blue cells and three pink cells form a ♡♡ pair.

Filter	Transformation	Blue set	Pink set	Agreement of p_1, p_2
F_0	0° rotation	$\{2,9,4\}$	$\{6,1,8\}$	15, 101
F_{90}	90° rotation	$\{2,7,6\}$	$\{4,3,8\}$	15, 89
F_{180}	180° rotation	$\{6,1,8\}$	$\{2,9,4\}$	15, 101
F_{270}	270° rotation	$\{4,3,8\}$	$\{2,7,6\}$	15, 89
F_{V}	vertical reflection	$\{2,9,4\}$	$\{6,1,8\}$	15, 101
F_{H}	horizontal reflection	$\{6,1,8\}$	$\{2,9,4\}$	15, 101
F_{D}	main diagonal reflection	$\{2,7,6\}$	$\{4,3,8\}$	15, 89
F_{A}	anti-diagonal reflection	$\{4,3,8\}$	$\{2,7,6\}$	15, 89

The ♡♡ property holds for all eight transformations. The resulting sets collapse into only two patterns: the row pair ($\{2,9,4\}$ and $\{6,1,8\}$, with $p_2 = 101$) and the column pair ($\{2,7,6\}$ and $\{4,3,8\}$, with $p_2 = 89$).

Observation 3.2.1. The 3-3 soaisu ♡♡ structure of the 3×3 magic square is stable under all eight symmetry transformations of the square (4 rotations and 4 reflections). That is, no matter how the filter is rotated or reflected, the blue cells and the pink cells always form a 3-3 soaisu ♡♡.

4. A Deep Connection: The 3×3 Magic Square and 3-3 Soaisu ♡♡

4.1. Systematic Generation via Linear Maps

In the previous section, we saw that the 3×3 magic square contains, as specific pairs of rows and columns arranged symmetrically, 3-3 soaisu ♡♡. In this section we reveal a new fact: one can generate **infinitely many** 3-3 soaisu ♡♡using the rows of the 3×3 magic square.

Row 1	**2**	**9**	**4**	**Row 2**	**7**	**5**	**3**	**Row 3**	**6**	**1**	**8**

The mechanism is simple. We use the three numbers in each row of the magic square as "coefficients," multiplying them by a, b, c respectively and summing the products. For example, from Row 1 $(2, 9, 4)$ and (a, b, c) we form the number $2a + 9b + 4c$. We perform this operation for each row, and then do the same after reading each row in reversed order.

Concretely, reading each row **left to right** gives:

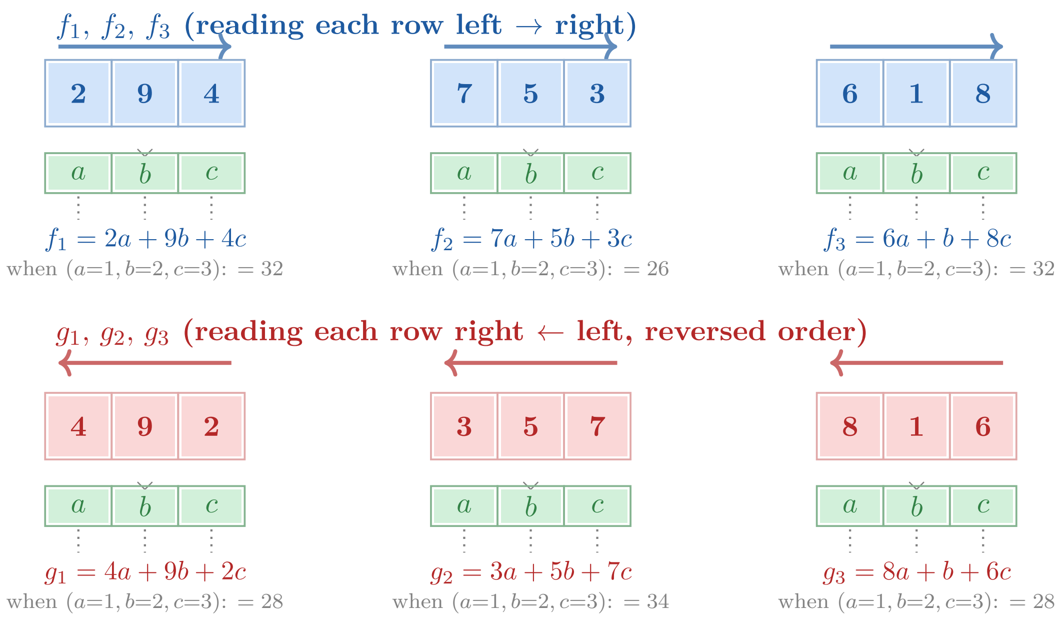
$$f_1(a,b,c) = 2a + 9b + 4c, \qquad f_2(a,b,c) = 7a + 5b + 3c, \qquad f_3(a,b,c) = 6a + b + 8c,$$

and reading each row **right to left** (in reversed order) gives:

$$g_1(a,b,c) = 4a + 9b + 2c, \qquad g_2(a,b,c) = 3a + 5b + 7c, \qquad g_3(a,b,c) = 8a + b + 6c.$$

The f_i are "weighted sums" reading the row in forward order, and the g_i in reversed order; the operation is akin to the inner product familiar from high school mathematics.

We illustrate this correspondence in the diagram below (blue: numbers from the magic square; green: a, b, c).



Theorem 4.1.1 (Infinite generation of 3-3 soaisu ♡♡)**.** *Let $S_1 = \{f_1(a,b,c),\ f_2(a,b,c),\ f_3(a,b,c)\}$ and $S_2 = \{g_1(a,b,c),\ g_2(a,b,c),\ g_3(a,b,c)\}$. For any real numbers a, b, c, the multisets S_1 and S_2 form a 3-3 soaisu ♡♡, possibly trivial:*

$$\sum_{x \in S_1} x = \sum_{x \in S_2} x = 15(a+b+c), \qquad \sum_{x \in S_1} x^2 = \sum_{x \in S_2} x^2.$$

Proof. **Equality of first-power sums.**

$$f_1 + f_2 + f_3 = (2+7+6)a + (9+5+1)b + (4+3+8)c = 15a + 15b + 15c,$$
$$g_1 + g_2 + g_3 = (4+3+8)a + (9+5+1)b + (2+7+6)c = 15a + 15b + 15c.$$

The coefficient 15 arises because the **sum of each column** of the magic square equals 15 ($2+7+6 = 9+5+1 = 4+3+8 = 15$).

Equality of second-power sums. Expanding, we find

$$f_1^2 + f_2^2 + f_3^2 = g_1^2 + g_2^2 + g_3^2 = 89a^2 + 107b^2 + 89c^2 + 118ab + 154ac + 118bc.$$

The equality of the coefficients of a^2 and c^2 (both equal to $89 = 4 + 49 + 36 = 16 + 9 + 64$) follows from the fact that f_i and g_i use the coefficients of each row in reversed order. □

Remark 4.1.2 (Extensibility of Theorem 4.1.1)**.** This theorem is not specific to the 3×3 magic square, and it is natural to ask whether it extends to larger magic squares.

For example, consider the following **associative magic square** [7] — a magic square in which every pair of entries symmetric about the center sums to $n^2 + 1$ —

$$M_4 = \begin{pmatrix} 1 & 8 & 12 & 13 \\ 14 & 11 & 7 & 2 \\ 15 & 10 & 6 & 3 \\ 4 & 5 & 9 & 16 \end{pmatrix}.$$

We verified algebraically that the linear maps formed from the rows of M_4 and their reversed-order counterparts generate a 4-4 soaisu of strength 2 (♡♡) for any real numbers (a, b, c, d); strength 3 does not hold identically for arbitrary real inputs.

The fact that both the order-3 and order-4 associative magic squares yield strength exactly 2 is unlikely to be a coincidence, but whether strength-2 soaisu holds for all associative magic squares of arbitrary order remains an interesting open problem.

4.2. A Concrete Example: $(a, b, c) = (1, 2, 3)$

Substituting $a = 1$, $b = 2$, $c = 3$: multiplying the magic square entries by a, b, c in each block and summing gives the following.

$$S_1:\quad \begin{array}{c}\mathbf{2}\\\mathbf{9}\\\mathbf{4}\end{array} \times \begin{array}{c}\mathbf{1}\\\mathbf{2}\\\mathbf{3}\end{array} \left.\begin{array}{c}2\\18\\12\end{array}\right\} 32 \quad \begin{array}{c}\mathbf{7}\\\mathbf{5}\\\mathbf{3}\end{array} \times \begin{array}{c}\mathbf{1}\\\mathbf{2}\\\mathbf{3}\end{array} \left.\begin{array}{c}7\\10\\9\end{array}\right\} 26 \quad \begin{array}{c}\mathbf{6}\\\mathbf{1}\\\mathbf{8}\end{array} \times \begin{array}{c}\mathbf{1}\\\mathbf{2}\\\mathbf{3}\end{array} \left.\begin{array}{c}6\\2\\24\end{array}\right\} 32$$

$$S_2:\quad \begin{array}{c}\mathbf{4}\\\mathbf{9}\\\mathbf{2}\end{array} \times \begin{array}{c}\mathbf{1}\\\mathbf{2}\\\mathbf{3}\end{array} \left.\begin{array}{c}4\\18\\6\end{array}\right\} 28 \quad \begin{array}{c}\mathbf{3}\\\mathbf{5}\\\mathbf{7}\end{array} \times \begin{array}{c}\mathbf{1}\\\mathbf{2}\\\mathbf{3}\end{array} \left.\begin{array}{c}3\\10\\21\end{array}\right\} 34 \quad \begin{array}{c}\mathbf{8}\\\mathbf{1}\\\mathbf{6}\end{array} \times \begin{array}{c}\mathbf{1}\\\mathbf{2}\\\mathbf{3}\end{array} \left.\begin{array}{c}8\\2\\18\end{array}\right\} 28$$

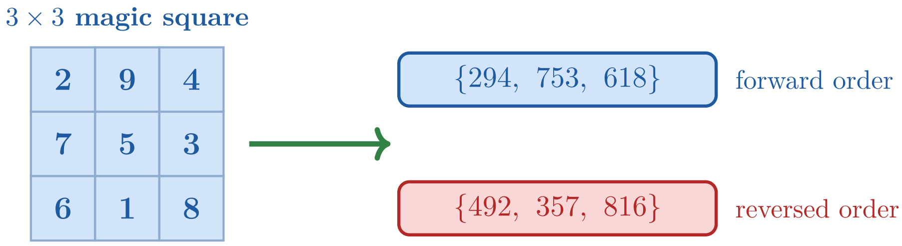
We obtain $S_1 = \{32, 26, 32\}$ and $S_2 = \{28, 34, 28\}$, with

$$32 + 26 + 32 = 28 + 34 + 28 = 90, \qquad 32^2 + 26^2 + 32^2 = 28^2 + 34^2 + 28^2 = 2724.$$

4.3. A Special Case: $(a, b, c) = (100, 10, 1)$

Substituting $(a, b, c) = (100, 10, 1)$, the values of f_i and g_i coincide with the three-digit decimal integers obtained by reading each row of the magic square:

$$S_1 = \{294,\ 753,\ 618\}, \qquad S_2 = \{492,\ 357,\ 816\}.$$

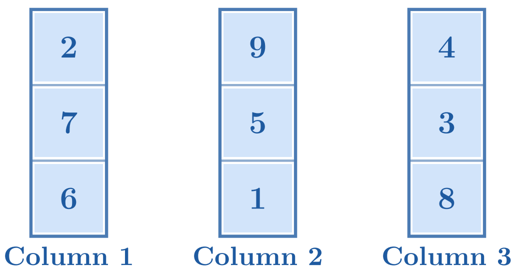


$$294 + 753 + 618 = 492 + 357 + 816 = 1665,$$
$$294^2 + 753^2 + 618^2 = 492^2 + 357^2 + 816^2 = 1{,}035{,}369.$$

As a consequence of Theorem 4.1.1, the "three numbers in forward order" and the "three numbers in reversed order" obtained by reading each row of the magic square as a three-digit integer automatically form a 3-3 soaisu ♡♡.

The essential appeal of this construction lies in the fact that **there is no restriction whatsoever on the choice of input** (a, b, c). Whether integers, rationals, or irrationals, substituting any values automatically yields a 3-3 soaisu ♡♡ pair.

We have so far used the **rows** of the magic square as coefficients, but the same construction using the **columns** as coefficients likewise produces 3-3 soaisu ♡♡ in exactly the same way.



That is, taking Column 1 $(2, 7, 6)$, Column 2 $(9, 5, 1)$, and Column 3 $(4, 3, 8)$ as coefficients,

$$h_1(a, b, c) = 2a + 7b + 6c, \qquad h_2(a, b, c) = 9a + 5b + c, \qquad h_3(a, b, c) = 4a + 3b + 8c,$$

and reading each column in reversed order,

$$k_1(a, b, c) = 6a + 7b + 2c, \qquad k_2(a, b, c) = a + 5b + 9c, \qquad k_3(a, b, c) = 8a + 3b + 4c,$$

the multisets $T_1 = \{h_1, h_2, h_3\}$ and $T_2 = \{k_1, k_2, k_3\}$ also form a 3-3 soaisu ♡♡ for any a, b, c. We encourage the reader to verify this by following the same steps as in the proof of Theorem 4.1.1.

5. Further Directions

This paper has illuminated the soaisu structure inherent in the order-3 grid — both the magic square and the natural square — from three perspectives. First, we reframed the fact that the outermost row and column pairs of the 3×3 magic square form 3-3 soaisu ♡♡ within the "soaisu filter" framework, and showed that this structure is invariant under all eight symmetry transformations of D_4. Second, we confirmed that domino tilings of the 3×3 natural square naturally produce a 4-4 soaisu ♡♡♡, an ideal solution of the PTE problem. Third, we proved algebraically that the linear maps defined by the rows (or columns) of the 3×3 magic square automatically generate 3-3 soaisu ♡♡ for any real input, and confirmed the possibility of extension to order-4 associative magic squares. These results demonstrate that soaisu is a powerful tool for analyzing the combinatorial structure of such squares.

Yet many questions remain to be resolved.

To begin with, the study of magic squares itself still harbors many open problems. The 3×3 magic square that is the subject of this paper is essentially unique up to rotation and reflection, but there are 880 distinct order-4 magic squares and 275,305,224 of order 5. For the order-6 magic square, the total count had long been undetermined, with only estimates reported, but in 2024 a computation by Hidetoshi Mino[4] finally established the exact count to be

$$17{,}753{,}889{,}197{,}660{,}635{,}632.$$

In this situation, where even determining the total number of magic square patterns of a given size is far from easy, the soaisu approach may offer a new perspective for dissecting and classifying the structure of magic squares.

For larger magic squares of order 4 and beyond, there is no shortage of questions to explore: whether the strength pattern of soaisu formed by pairs of rows, columns, and diagonals is invariant under D_4 transformations; what soaisu structures emerge from domino tilings; and many more.

Moreover, soaisu is a rich concept with connections not only to magic squares but also to diverse mathematical contexts such as number theory, combinatorics, group theory, and linear algebra (in particular, determinants)[3]. We look forward to further research entering this still largely uncharted territory.

Acknowledgment

The author is grateful to the anonymous referee and to the editor for their careful reading and valuable comments, which have improved the presentation of this paper. The author also thanks Claude (Anthropic) for assistance with LaTeX typesetting, TikZ figures, English translation, and Python/SymPy verification of the computations presented in this work.

Funding

The author received no specific funding for this work.

Conflict of Interest

The author declares that there is no conflict of interest regarding the publication of this paper.

Data Availability Statement

No data were used to support this study.

References

[1] C. Shuwen, *A survey of the Prouhet–Tarry–Escott problem and its generalizations*, arXiv:2506.11429 [math.NT], 2025. https://doi.org/10.48550/arXiv.2506.11429

[2] P. Borwein, P. Lisoněk and C. Percival, *Computational investigations of the Prouhet–Tarry–Escott problem*, Mathematics of Computation, **72(244)** (2003), 2063–2070. https://doi.org/10.1090/S0025-5718-02-01504-1

[3] K. Takemura, *The magic of a 12-12-12 split: discovering soaisu*, Math Horizons, **33(3)** (2026), 21–23. https://doi.org/10.1080/10724117.2025.2578498

[4] H. Mino, *Number of magic squares of order 6*, OEIS A006052, 2024. https://oeis.org/A006052

[5] K. Takemura, *Algebraic symmetries and the 90-degree rotation complement theorem in domino tilings of order 4 natural squares*, Utilitas Mathematica, **127** (2026), 183–208. https://doi.org/10.61091/um127-13

[6] K. Takemura, *The beautiful soaisu partition*, Chalkdust Magazine, Issue 23, 2026. https://chalkdustmagazine.com/features/the-beautiful-soaisu-partition/

[7] E. W. Weisstein, *Associative Magic Square*, MathWorld—A Wolfram Web Resource. https://mathworld.wolfram.com/AssociativeMagicSquare.html