

Tessellations from hyperplane families: Weyl and kagome cases

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Abstract

For a research project with colleagues, I have been studying tessellations of Euclidean spaces which arise from families of hyperplanes. Many examples come from a finite type root system and associated finite Weyl group, W , whose affine extension acts on the tessellation. Other examples do not come from a root system and Weyl group. In this talk I will discuss the details of some Weyl and some non-Weyl kagome tessellations, including 3D printed models which help to visualize the geometry.

Introduction and Definitions

A mirror reflection is the simplest kind of symmetry, and a group generated by reflections, r_i , $i = 1, \dots, n$, with $r_i^2 = I$, satisfying relations

$$(r_i r_j)^{m_{ij}} = I, \quad 1 \leq i < j \leq n,$$

is called a **Coxeter group**. In $\mathbb{R}^n$ with the standard dot product, any nonzero vector, α , determines a reflection

$$r_\alpha(\beta) = \beta - \frac{2\beta \cdot \alpha}{\alpha \cdot \alpha} \alpha \quad \text{for} \quad \beta \in \mathbb{R}^n$$

satisfying $r_\alpha(\alpha) = -\alpha$, $r_\alpha^2 = I$ and whose mirror is the fixed hyperplane

$$H_\alpha = \{\beta \in \mathbb{R}^n \mid \beta \cdot \alpha = 0\}.$$

Among the Coxeter groups are the **Weyl groups** which are generated by reflections r_α for α in a **root system** Φ .

A root system is a subset Φ of $\mathbb{R}^n$ satisfying:

- (1) Φ spans $\mathbb{R}^n$ and does not contain 0.
- (2) For $\alpha \in \Phi$ the only multiples of α which are in Φ are $\pm\alpha$.
- (3) For $\alpha \in \Phi$ the reflection r_α leaves Φ invariant.
- (4) For $\alpha, \beta \in \Phi$ we have the **Cartan integer** $2\frac{\alpha \cdot \beta}{\beta \cdot \beta} \in \mathbb{Z}$.

The Weyl group W associated with Φ is the group generated by the reflections r_α for all $\alpha \in \Phi$. There is a minimal generating set, $\Delta = \{\alpha_1, \dots, \alpha_n\}$, of such α called **simple roots**, and W is a Coxeter group where the exponents m_{ij} in the relations are determined by the $n \times n$ **Cartan matrix**

$$C = [c_{ij}] = \left[2 \frac{\alpha_i \cdot \alpha_j}{\alpha_j \cdot \alpha_j} \right].$$

When $c_{ij}c_{ji}$ equals 0, 1, 2, 3 then m_{ij} equals 2, 3, 4, 6, respectively.

For each $\alpha \in \Phi$ and each $k \in \mathbb{Z}$ we have the parallel hyperplanes

$$H_{\alpha,k} = \{\beta \in \mathbb{R}^n \mid \alpha \cdot \beta = k\} = H_{-\alpha,-k}$$

which give a finite number of families of parallel hyperplanes. Let

$$\mathcal{H} = \{H_{\alpha,k} \mid \alpha \in \Phi, k \in \mathbb{Z}\}.$$

Each of these hyperplanes is the fixed point mirror of a reflection isometry

$$r_{\alpha,k}(\beta) = \beta - (\alpha \cdot \beta - k)\alpha \quad \text{for} \quad \beta \in \mathbb{R}^n$$

and the group generated by all these reflections is the **affine** Weyl group, $\widetilde{W}$, an infinite extension of W whose structure is known to be a semidirect product of W with a group of translations of $\mathbb{R}^n$.

Each hyperplane is the boundary of a pair of half-spaces

$$H_{\alpha,k}^{\pm} = \{\beta \in \mathbb{R}^n \mid \pm(\alpha \cdot \beta - k) \geq 0\}$$

A fundamental domain for the action of W on $\mathbb{R}^n$ is the polytope **alcove**

$$A_{finite} = \{\beta \in \mathbb{R}^n \mid \alpha_i \cdot \beta \geq 0, \ 1 \leq i \leq n\} = \bigcap_{1 \leq i \leq n} H_{\alpha_i,0}^+.$$

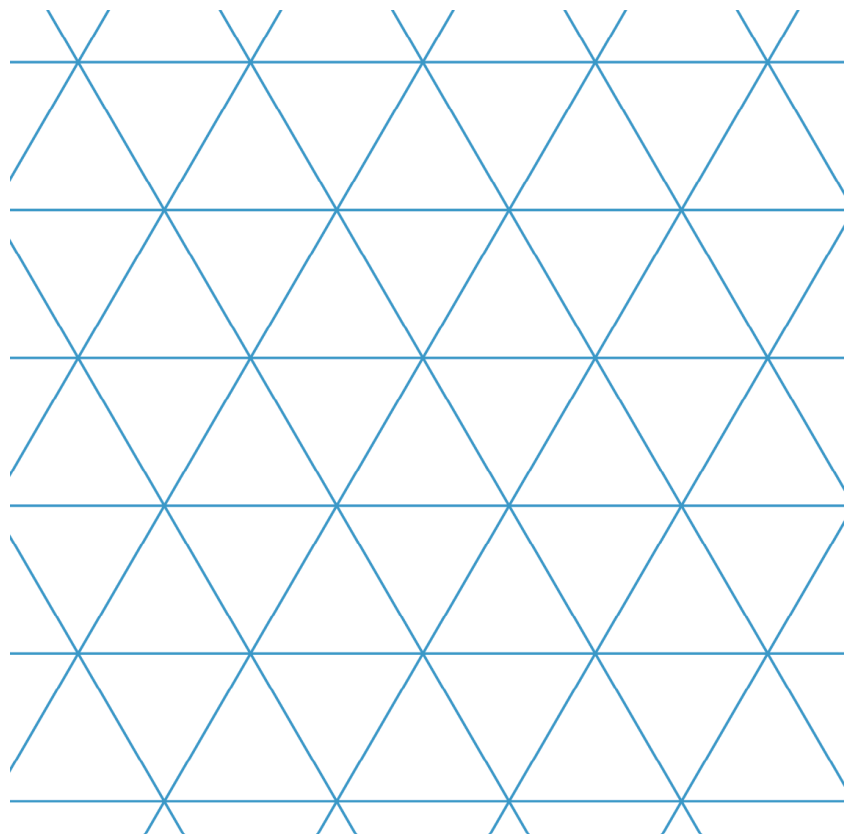
A fundamental domain for the action of $\widetilde{W}$ on $\mathbb{R}^n$ is the **tile**

$$A_{affine} = A_{finite} \cap H_{\theta,1}^-$$

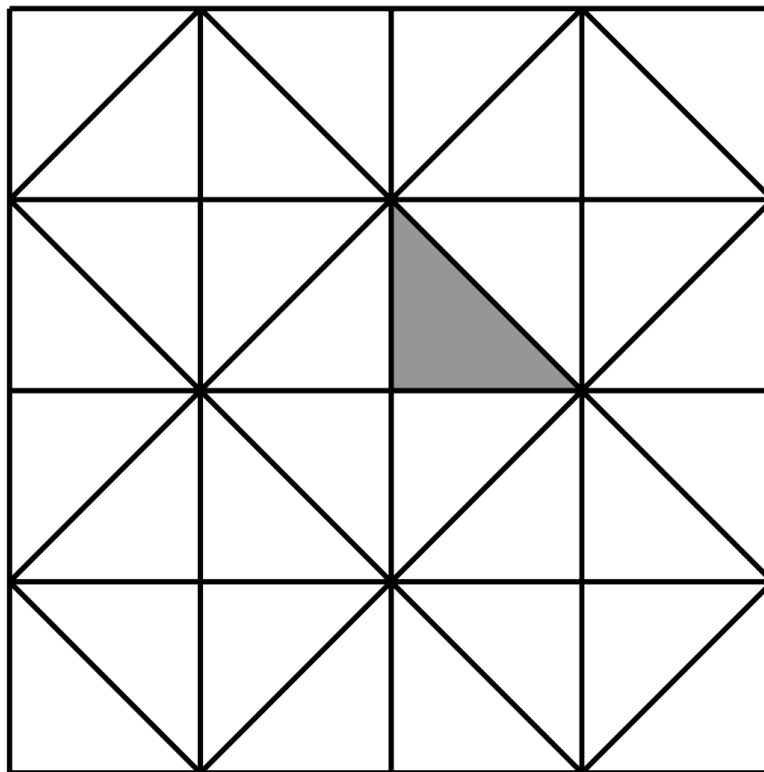
where $\theta \in \Phi$ is the **highest root** with respect to the choice of simple roots, Δ .

Let $\overline{V} = \mathbb{R}^n \setminus \bigcup_{H \in \mathcal{H}} H$ and note that its connected components are each isometric to the interior of A_{affine} , and $\widetilde{W}$ acts simply transitively on these tiles. The tessellation of $\mathbb{R}^n$ is the decomposition of $\mathbb{R}^n$ by $\mathcal{H}$ into tiles. We may also discuss structures within this decomposition such as the intersection of faces of tiles, layers between parallel hyperplanes in $\mathcal{H}$, and the intersections of such structures.

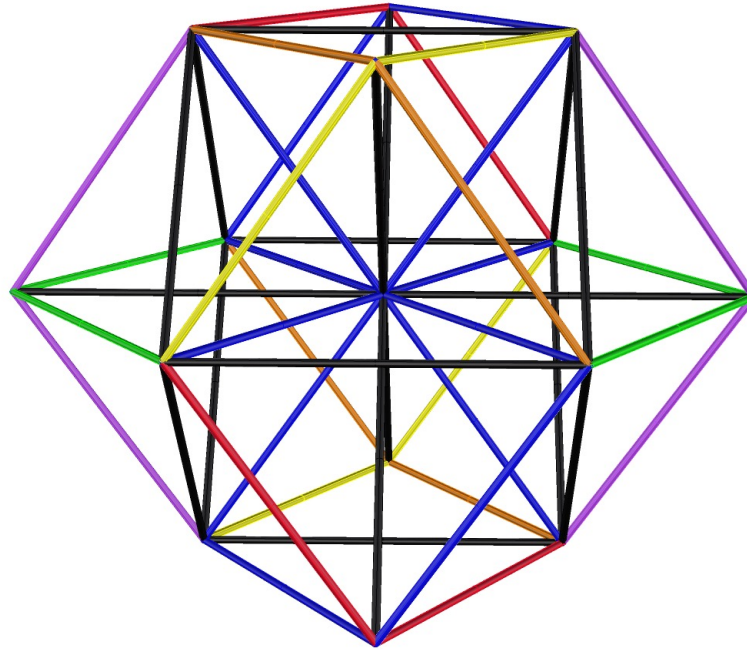
In the simplest case of $n = 1$ where $\mathbb{R}$ is tessellated by unit interval tiles, $W = S_2$ and $\widetilde{W} = D_\infty$ is the infinite dihedral group. In one $n = 2$ case $\mathbb{R}^2$ is tessellated by equilateral triangles and $W = S_3$. In another $n = 2$ case $\mathbb{R}^2$ is tessellated by $45 - 45 - 90$ triangles. In that first case the root system, $\Phi = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 + \alpha_2)\}$, is of type A_2 with Cartan matrix $C = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$. In the second case, $\Phi = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 + \alpha_2), \pm(\alpha_1 + 2\alpha_2)\}$, is of type B_2 with Cartan matrix $C = \begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix}$.



Tessellation of affine type A_2



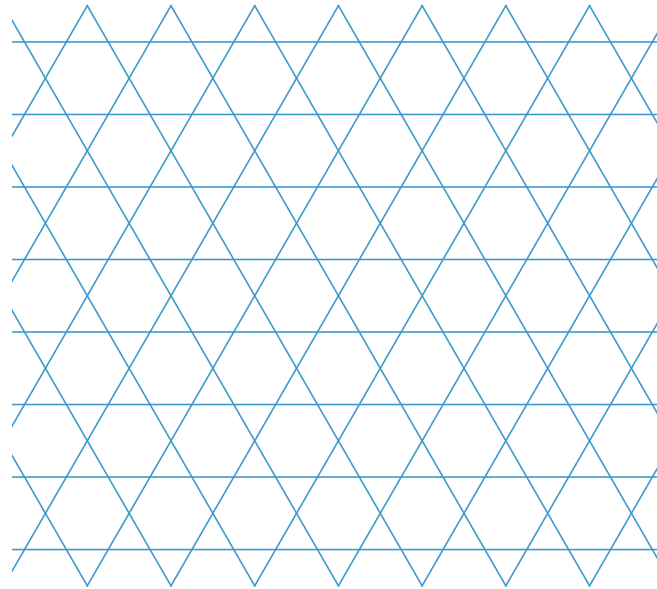
Tessellation of affine type B_2



Tessellation of affine type A_3
24 Tetrahedra form a rhombic dodecahedron

The 2D kagome tessellation

The 2D kagome tessellation is made from two kinds of tiles, equilateral triangles and regular hexagons. There are three families of parallel lines as there are in the affine A_2 tessellation, at angles 0, 60 and 120 degrees, but in this 2D kagome tessellation those lines are spaced further apart.



2D Kagome tessellation

The 2D kagome tessellation can be refined to the affine A_2 tessellation by adding lines in each family which are midway between each pair of adjacent parallel lines. The added lines pass through the centers of the hexagons, subdividing each hexagon into six triangles. Starting with the affine A_2 tessellation with each line in a family labelled by an integer, we obtain the 2D kagome tessellation by deleting in each family of parallel lines the lines labelled by an even integer. Translations along a line of the tessellation by twice the length of an edge will preserve the tessellation, so a fundamental domain for the translation subgroup of isometries is a parallelogram consisting of one hexagon and a pair of triangles attached to the hexagon on opposite edges.

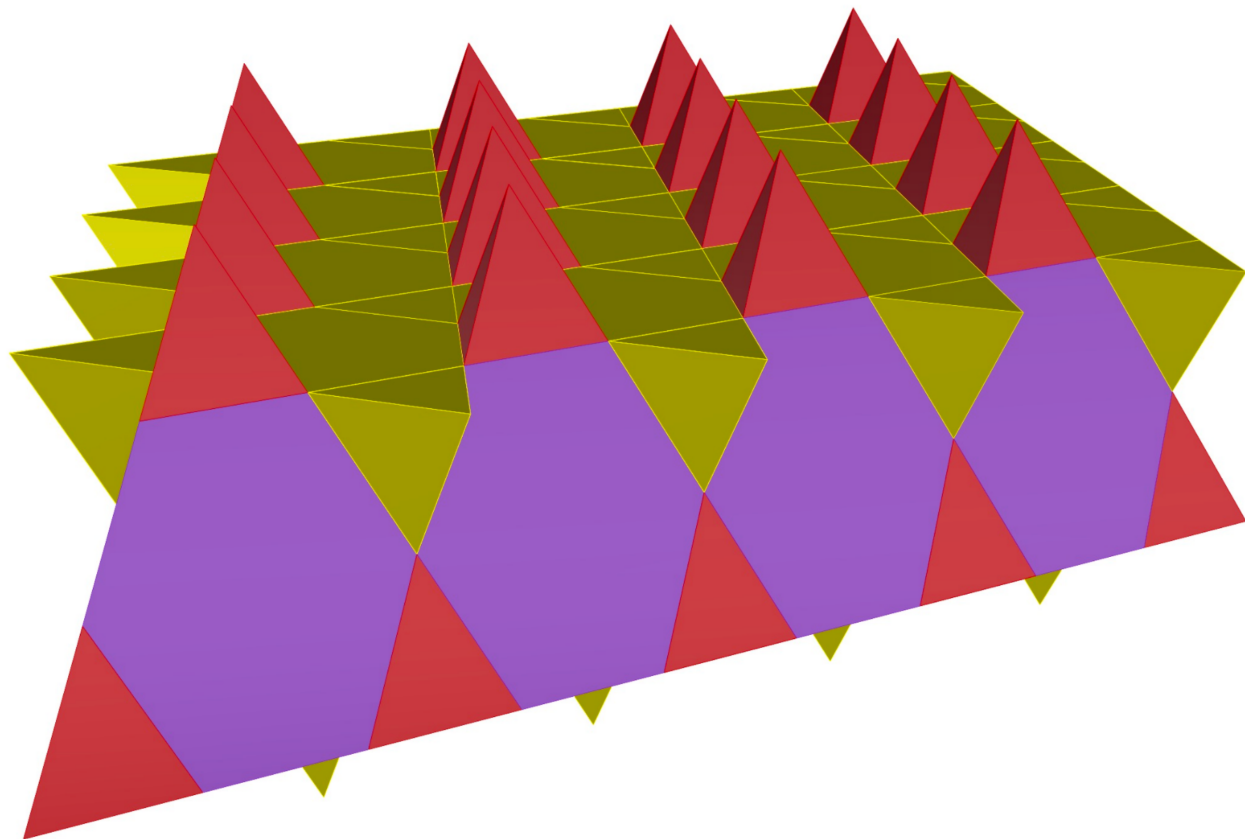
For this tessellation the interesting substructures are finite sets of tiles, layers of tiles between parallel lines, and alcoves of tiles bounded by lines at 60 degrees.

The 3D kagome tessellation (quarter cubic honeycomb)

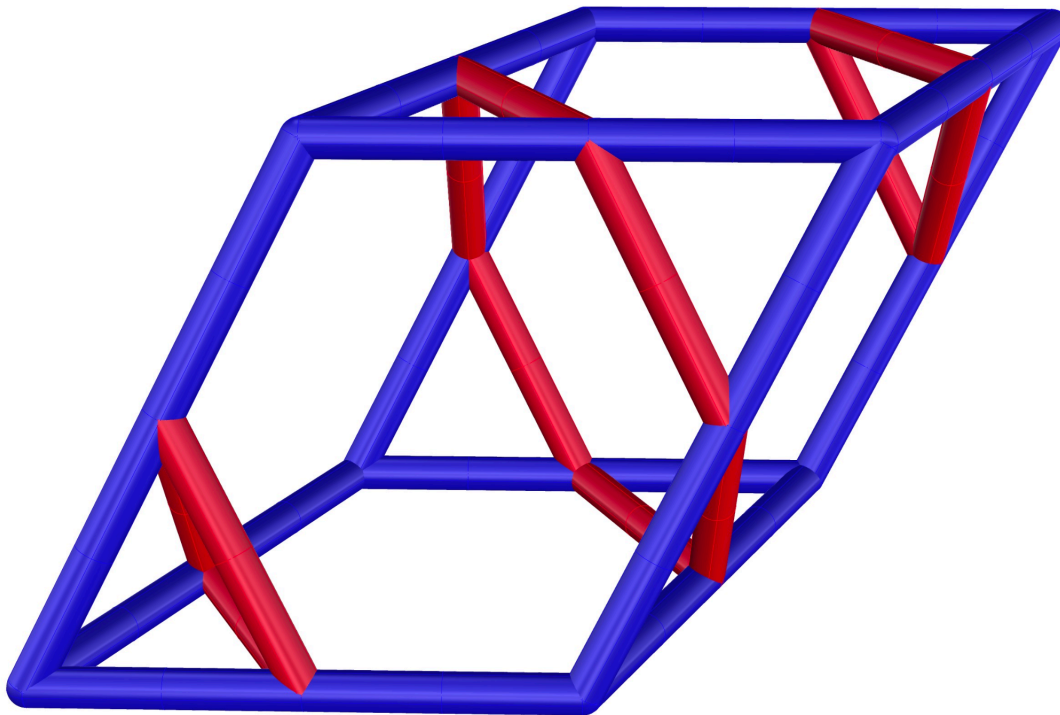
To get the 3D kagome tessellation take a regular tetrahedron, T , with edge length 3 and mark on each edge two points which divide the edge into three equal segments of length 1. For each of the four vertices, v_i , $1 \leq i \leq 4$, of T , form a regular small tetrahedron, ST_i , with edge length 1 whose other three vertices, v_{ij} , $1 \leq j \leq 3$ are the edge points marked on the edges incident to v_i closest to v_i . The convex hull of the 12 vertices, v_{ij} , $1 \leq i \leq 4$, $1 \leq j \leq 3$, is a **truncated tetrahedron**, TT , all of whose edges have length 1. Four faces, TF_i , of TT are the triangles with vertices, v_{ij} , $1 \leq j \leq 3$, for a fixed choice of i . There are also four hexagonal faces, HF_i , of TT , each parallel to one of those triangular faces. Translation by length $2n$ for any integer n along any edge of TT will move TT to another copy of it so that the spaces between copies are small tetrahedra of edge length 1, some pointing up and some pointing down.

The planes formed by the triangular and hexagonal faces of TT come in four families of parallel planes, whose normal vectors are the four heights of T . The picture below, made in Rhino3D, shows several copies of TT in blue, as well as copies of ST_i in red pointing up and some in green pointing down. To tessellate all of $\mathbb{R}^3$ with copies of TT and ST , we should reflect T by its bottom face, rotate it by 60 degrees around a vertical axis, and then translate it by 2 along a non-horizontal edge of T . That gives the downward pointing copies of T shown in the picture below in green. Note that each plane of this tessellation comes with a 2D kagome tessellation.

In addition to the static pictures in this document, I can also show 3D printed models showing interesting features of this tessellation.

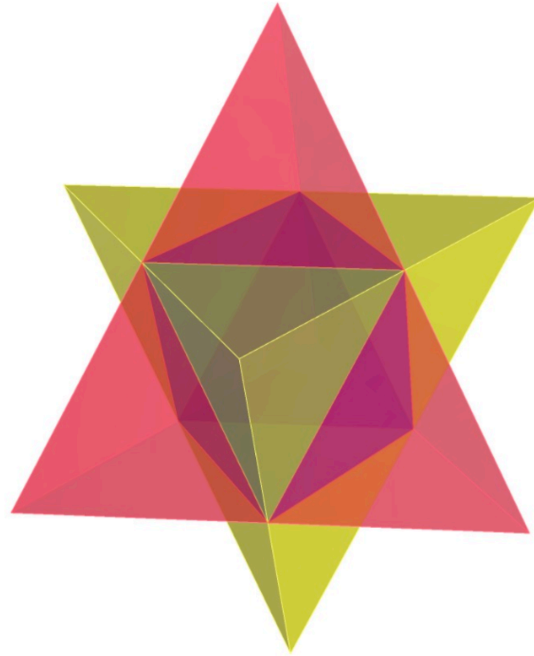


3D kagome tessellation



3D kagome fundamental domain under translations

Another tessellation of $\mathbb{R}^3$ by regular octahedra and tetrahedra is obtained from the 3D kagome example just by shifting the red planes. Here is a picture of one octahedron surrounded by eight tetrahedra attached to its faces.



3D TetraOctahedral tessellation

Thanks for your kind attention!